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Nonbinary Information Transmission

By

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Reprinted from the "Proceedings of The Institution of Radio and Electronics Engineers Australia"
Vol. 30, No. 12, Pages 377-479, December, 1969

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CONTENTS

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Manuscript received by The Institution December 14, 1966.
U.D.C. number 621.391.

4

CONTENTS (Continued)

Part I—Elements of Nonbinary Systems

A. Introduction

Binary information handling is already so well known that children learn about in in grammar school. Indeed, what could be simpler than expressing information in yes-no or true-false statements? Digital computers and many data transmission systems operate in binary modes. The advantages in circuit reliability, simplicity of testing equipment, and precision in calculations are obvious. The principal disadvantage is the unnecessarily large number of individual signal elements which the equipment must handle to perform a given task. This is particularly important in commercial communications, where the customer pays a fixed amount per time unit for the use of a channel of given bandwidth, such as an ordinary telephone channel. To shorten the time interval of occupying the channel, the user is interested in transmitting the largest amount of information per time unit. If the information can be expressed in numbers, the user will learn that he needs four times as many elements to transmit his numbers in binary elements than he would need could he transmit the numbers directly in decimal digits (i.e., in nonbinary modes). If the information is in continuous waveforms such as speech signals and if it has to be transmitted in real time, i.e., without appreciable delay, the user will learn that the binary mode will require up to 15 times the bandwidth of the continuous mode. The communications carrier will charge higher rates for the channel with the wider bandwidth. In examples like these, the question arises: would nonbinary modes of operation be more economical? And if so, at which base (number of symbols in the nonbinary alphabet) is the optimum tradeoff between the improved performance of the nonbinary equipment and its increased complexity and cost? It is the object of this paper to review the progress in the development of nonbinary systems which has been made during the last two decades in many countries. It will be shown that the mathematical and graphical tools are now available to perform such tradeoff analyses.

The history of information transmission technology shows that nonbinary systems actually preceded binary (Cherry 51). In 1823 Ronalds used a single wire with the ground as return circuit, over which he transmitted letters of the alphabet by closing the circuit in one out of 26 subintervals of the total interval reserved for each character. This is clearly a quantized pulse position modulation (PPM) system. In 1832 S. F. B. Morse introduced his famous dot-dash code with 5 symbols: dot, dash, and 3 different lengths of gaps. It was not until 1853 that Whitehouse used, for the first time, the highly efficient 5-unit telegraphy code, the first binary transmission method (Filipowsky 55).

Nyquist (24) and Hartley (28) most likely represent the first steps toward a theoretical analysis of nonbinary systems They defined as information (in the notation of the present paper) the expression:

$$R_{NH} = k \cdot \log_2 m \quad , \tag{1}$$

which indicates that the information content of a word consisting of k characters in a row (ordered sequence) is proportional to k and to the logarithm of m, the number of symbols in the alphabet from which each character can be chosen. The basis of the logarithm determines the unit of information. The basis 2 (Hartley 28) establishes the familiar binary unit, called one bit. This fundamental relationship indicates that a nonbinary system of m symbols in the signal alphabet can transmit $\log_2 m$ times more bits per word than a binary system with the same number of elements per word. Later investigations by Tuller (49) and Shannon (48) established the relations between the transmission rate R_B, the channel capacity in the presence of noise, and the entropy of the information flow. These relations took two important non-deterministic aspects of digital transmission systems into consideration: the uncertainty of correct reception of signals and the probabilistic characteristics of the choice of "letters" when forming "words" for transmitting messages.* Probability theory and statistical mathematics were correctly brought into the theory of digital transmission systems. At the same time (1947) V. A. Kotel'nikov in Russia submitted his doctoral dissertation, which was later translated into English (Kotel'nikov 59). Unfortunately, it had been virtually unknown outside Russia until about 1956. This work contains a special chapter on the noise immunity for signals with many discrete

*The terms "letter" and "word" refer to any symbol or group of symbols, not necessarily to elements of written or printed languages.

values, stressing the advantage of telegraphy using 32 orthogonal signals—one for each letter of the alphabet and for some symbols. This is clearly a nonbinary approach.

All these pioneers of communications theory concluded that the extraction of the information-carrying signals from noise can best be performed when the signals are relatively long in terms of their reciprocal bandwidth. The more recent theory speaks of signals with a large time-bandwidth product (also called TB product or WT product) or of signals of high dimensionality (Rice 50). If such large TB systems run also at a high transmission rate, one has only one approach : to let each signal carry a large amount of information. Only nonbinary systems can achieve this goal as evidenced by equation 1. Nonbinary systems are also more suitable than binary systems to fulfil the requirements for integrated data systems (Filipowsky 59). These are systems which combine an optimum waveform design with a number of additional improvements, such as error correction codes, redundancy reduction, and communications feedback.

The work of Middleton and Van Meter (55a) established the decision-theoretical background for optimum multi-alternative detection schemes. This and many related papers thereafter lead to the definition of mathematical models for optimum receivers of nonbinary signals. Section D will give additional references about related analytical investigations.

Since 1955, practical designers of digital communications systems have become interested in nonbinary systems. Almost 400 papers have been published about the theory and the practice of this new class of communications systems. It is the object of this review paper to show how the scores of different methods can be grouped and how closely, in this special science, theory and practice support each other. A particular tool has been developed to compare numerically the achievements of the various nonbinary approaches. Called the utility chart, it is an original contribution by this author. Its value will become evident to the reader throughout this paper, primarily in the many cases where practical measurements on operational systems will be compared with ideal mathematical models.

B. Transmission Problems for Digital Systems

Transmission media used for nonbinary data transmission include practically all media which can carry any type of communications signals, most of all telephone circuits, high-frequency radio links, scatter links, and (recently) space communications links.

B1. *Transmitter Limitations*

The transmitting equipment of any data system generates the information-carrying signals, encodes the input messages into these signals, and matches the signals to the constraints of the channel. The latter task imposes the principal limitations. The signal power must be limited to a prescribed average power or peak power and the power spectrum of the signals should have no components outside the prescribed transmission band B. In most data transmission systems the signals are, by definition, confined to a fixed time interval, T, and the mathematical function representing the signals is assumed to be identically zero outside this interval.

Nyquist (28) and Gabor (46) show that both the last conditions cannot possibly be fulfilled simultaneously. Many recent papers made the problem of bandlimited signals with small TB the subject of their investigation. Gibby and Smith (65) extended Nyquist's original theory to transmission systems with gradual frequency cutoff and to band-pass systems with non-ideal filter characteristics. Rubin and DiFranco (63) developed an analytical representation of time-limited signals which can be easily applied to rectangular pulses.

Another important transmitter limitation may result from the sequential combination of elementary signals. The spectrum of a pair of signals is usually different from the spectrum of either signal (Pushman 63). The influence of discontinuities between adjacent elementary signals can be of great importance in multiphase shift keying (MPSK) systems. Karshin (65) gives a table of the effective TB for several sequences of PSK signals.

Most of the above-mentioned limitations become less important if the signal base (TB) can be increased. Larger sets of longer waveforms can be used in nonbinary systems without reducing the rate as compared with equivalent binary systems. Several such sets with larger TB are discussed in Part IV.

B2. *Channel Disturbances*

Noise and disturbances created in the transmission medium are the most detrimental effects preventing digital data transmission systems to achieve the theoretical transmission rate. The fact is, and we shall prove this point in section C, that most practical systems do not even reach ten percent of the theoretical rate. This fact has been pointed out many times in the past (see, for example, Filipowsky 59) and many investigations have been started to explore the reasons for this low " efficiency " of data transmissions systems. The results now clearly establish what many practical engineers call the " facts of life " : the mathematical models which are used to represent an ideal system assume an oversimplified type of noise, the so-called white Gaussian noise ; and they also assume a linear, constant transmission factor for the channel. The actual observations, as quoted below, do not support this assumption.

The observations of channel disturbances find an appreciable difference in the characteristics of at least three different classes of disturbances : pulsive noise in telephone circuits, atmospheric noise in HF radio circuits, and man-made noise in mobile radio equipment. A kind of natural disturbance which follows fairly well the characteristics of the Gaussian noise model apparently is the noise in space communications circuits. Circuit noise and fluctuation noise in electronic devices (also known as thermal agitation noise) likewise closely follow the Gaussian model. However, the improvements in low noise solid state devices reduced the importance of these noise sources when compared to noise sources in the transmission medium.

The characteristics of *pulsive noise in telephone circuits* have been observed either directly (Fennick and Nasell 66, Wainwright 61, Bodonyi, 61) or indirectly by the distribution of the errors which the noise causes in data transmission systems (O'Neil 65). Special methods have been devised for the performance of laboratory tests of data transmission equipment, using recordings of real noise taken from telephone circuits (Fennick 65). The results of all these observations, whether recorded in U.S.A., in Russia (Purtov et al 65) or in Europe (Ebenau and Stotesbury 64) are in good agreement. They indicate that the noise in telephone circuits consists basically of two components. One component, called the background noise or regular component, can be represented by a Gaussian distribution, but its average power is relatively small. The second irregular component consists of high spikes that occur relatively infrequently, but with apparently devastating effects on short binary signals. Any noise model that takes only the average power of such noise into consideration is apparently inadequate.

The characteristics of *atmospheric noise* are similar to the pulsive noise in telephone circuits. Atmospheric noise, too, can be represented by a non-Gaussian component, added to a Gaussian component (Shepelavey 63). The former may be represented, during short periods of time (a few minutes to an hour), by an exponential distribution (Spaulding 64). For systems below 20 MHz, the atmospheric noise is Rayleigh distributed at its lowest levels, but follows anoth

exponential law at higher levels. Results of an extensive program of experimental determination of the amplitude probability distribution (APD) of atmospheric noise have recently been published (Spaulding 66). The measurements covered carrier frequencies in the range from 13 kHz to 30 MHz with channel bandwidth from 0.6 Hz to 4 kHz. Averaging times from 10 minutes to more than 12 hours were used. The interesting result is that all APD with observation parameters covering these wide ranges in frequency, bandwidth, and integration time had the same form, and can be represented to a good degree of accuracy by a single mathematical description. Despite this commonality between various observations of atmospheric noise, the fine structure in the time domain may differ (Nakai 65) and, in general, will be different from the fine structure of telephone noise. An additional difference is due to the combined effect of atmospheric noise and fading (Conda 65, Bello 65b), a problem unknown in telephone channels.

Man-made noise is of particular importance in mobile radio channels where neither the transmission medium nor the receiving site can be protected against external electromagnetic influences. It is of less importance to data transmission, since most mobile radio links carry voice communications. This may change. Noise created by vehicles, radar, and other sources is already a significant problem in tactical army data transmission situations (Van Houten and Rowe 63).

Cross talk from one communications channel to another is a special kind of disturbance which may result from nonlinear distortions or from a number of other sources. It may assume the character of pulsive disturbances; indeed, some part of telephone noise is due to cross talk. In radio transmission, cross talk may assume the nature of a sinusoidal disturbance and in electromagnetic interference situations it may produce any kind of well-defined but highly disturbing waveforms. No special analysis of cross talk is advisable until a specific design and its electromagnetic environment has been defined.

B3. *Channel Distortions*

It may be assumed that the equipment designer will make every effort to avoid unintentional distortions of the transmission signals in transmitter, amplifiers or receiver. This leaves the systems engineer still with a large variety of distortions from the characteristics of the transmission medium. These distortions assume different character for at least four different transmission media: lines and cables, radio relay links (line-of-sight), long distance radio links (including short waves and all types of scatter channels) and finally space links. This subdivision parallels to some extent the subdivision of the channel disturbances. Line-of-sight links and space links had not been specifically discussed under channel disturbances, because they can be well represented by Gaussian noise models. Any other special features of these links which are of importance to nonbinary systems will be discussed here.

Lines and cables present the data transmission system designer with more difficult problems than the designer of telephone voice circuits. It is primarily the phase characteristic of telephone circuits which is of negligible importance to voice communications but is of very high importance to data transmission. Phase distortions (also called delay distortions) and frequency response (attenuation characteristics) of the telephone network of the Bell System in the United States are presented in 1960 in a paper by Alexander et al (60), with additional information given by Morris (62). This can be supplemented with information about the distortions measured over Russian (Purtov et al 65), German (Von Hänisch and Kettel 63), and British (K. L. Smith et al 62, Croisdale 61) telephone and telegraph facilities. There is general agreement that leased lines (Turrel 66) are more suitable for data transmission than the automatically switched telephone network (Townsend and Watts 64). This is understandable because leased lines can be handpicked while the selection of switched lines is made at random by an automatic device. It is also reasonable to apply some lengthy effort to balance and improve leased lines by manually adjusted equalizers while such action for switched connections requires automatically adaptive complex devices. Quite favourable conditions prevail for data transmission over transoceanic cables (Croisdale 61). Army facilities (Tucker and Duffy 63), and facilities for weapon systems (Cortizas and Wolff 64) have to provide wide safety margins for tolerating distortions to secure reliable operation under all operational adversities. Such military facilities impose the additional requirements that several generations of older and newer equipment have to be compatible with each other. This requirement makes the matching and balancing of lines rather undesirable.

In addition to the undesirable delay distortions there are three other peculiar distortions which affect data transmission much more than voice communications (Morris 62). *Drop-out*, i.e. line interruptions, of only a fraction of a second may cause bursts of hundreds of errors in a data system, while during a voice communication they may not be noticed at all or, in the worst case, require the repetition of a word. *Frequency translations* (Evans et al 61) occur primarily in single sideband carrier equipment of older vintage. They can have disastrous effects on data transmission equipment. In extreme cases it may happen that a different number of bits is received than has been transmitted or that the polarity changes from "mark" to "space" after going through a number of errors. Naturally, protective measures can be introduced against such disturbances. It is, however, imperative that the designer of a nonbinary transmission system is fully aware that such disturbances may occur in a particular facility. *Synchronization difficulties* can be caused by a number of line distortions, such as the above-mentioned frequency translations, or drop-out, but also by disturbing tones caused by faulty echo suppressors or operators' mistakes.

Radio relay links may create their own peculiar disturbances due to aircraft reflections, weather influences or even birds. Many years of experience have taught telephone companies how to safeguard the facilities against most of the disturbances. Automatic switching devices and complex supervisory systems protect installations against sudden failures and replace disabled units within seconds. Such "switchovers" can hardly be noticed during voice conversations while they cause catastrophic errors in data transmissions. The increasing demand for higher data rates will lead to the provisions of special wideband channels for data transmission. Best candidates are the 12-channel group, the 60-channel supergroup and the 600-channel master group. Radio relay links are now the principal carriers for these multiplex facilities and they surely will be used for the much simpler data links. A report by Suyderhoud (65) describes the data transmission characteristics of such wideband links when established over the L-multiplex system of the Bell telephone facilities in the U.S.A.

Long distance radio circuits have been in use for telegraphy, amateur services, aircraft communications and for military purposes since the beginning of this century. The heavy disturbances on short wave links (3 to 30 MHz) or on scatter links over tropospheric, ionospheric, or meteoric scatter media make them a challenge for data system designers. The present situation of HF (short wave) data transmission equipment is still far from satisfactory, though great progress has been made in the last 10 years (Goldberg 61, Greim et al 65, Brennan 58, Bello 65b); nonbinary systems in horizontally encoded frequency division multiplex modes may make further improvements possible, particularly when communications feedback is applied and when orthogonal signal sets are used (Tsikin 63). Fritchman and Leonard (65) and Fontaine (63) report the application of forward error control modes to troposcatter data links,

while the Federal Aviation Agency in the U.S.A. performed long distance VHF communications systems tests (De Zoute and Pearson 64). This valuable data of recent field tests shows the large gap which is still to be closed between theory and practical performance.

The longest distances in the universe have been traversed by data transmission between planetary deep space probes and earth. Such *space communications links* (Filipowsky and Muehldorf 65a, chapter 7) display special disturbances which are normally not present in other data transmission links. Frequency shifts due to *Doppler effect* have been the most feared disturbance to space data links (Tischer 59). Originally it was thought that the best countermeasure would be to frequency-modulate the radio frequency carrier with a base-band void of dc components and low frequency components. The popularity of FM-FM telemetry in the earlier space missions is partly due to its insensitivity to Doppler shifts. However, these early anti-Doppler measures are inefficient in the use of the available transmitter power and bandwidth. The requirement for precise tracking of spacecraft is another important reason for improved anti-Doppler measures. Digital correlation detection of Doppler shifted signals (Miller 64) is now generally applied for acquisition and locking-in signals from deep space probes. The method provides local oscillator signals in the Earth-based receivers, which track the Doppler shifted variable carrier frequency with the result that the intermediate frequency in the receiver is basically free from Doppler variations (Mathison 62).

All signals between spacecraft and Earth have to penetrate the Earth's atmosphere and distortions may occur on such paths. The nature and the intensity of these distortions and disturbances depend on the frequency range (Tischer 65). Nonbinary data systems with their more complex waveforms need careful attention to these propagation phenomena (Baghdady 65). Attenuation, polarization, and other effects of the interplanetary medium on S-band links are discussed by Easterling and Goldstein (65). Surveys of the effects of propagation effects on space data links are also available in a recent NASA report (Dabul 66) and in a Russian book on space radio communications in English translation (Petrovich and Kamnev 66).

Summarizing this section one can state that much has been learned in the last decade about the adverse effects of transmission channels on data transmission systems. Unfortunately, the conclusion is that the characteristics differ greatly in different media and, even in channels of the same type, the adverse effects vary with geographical location, season, daytime, and many other environmental factors. Data systems will need high flexibility and, most likely, adaptive modes of operation to meet these large variations in channel characteristics. Theoretical and practical indications are that nonbinary systems offer these desirable characteristics.

C. Performance Criteria

The merits and costs of several competing data systems can be compared only after performance criteria have been established. These criteria should be equally suitable for theoretical analysis, for laboratory measurements under simulated channel conditions, and for field tests under actual operational conditions.

C1. *Definitions*

There are at least six considerably different kinds of characteristics which have to be considered when judging the overall performance of any data system.

The first characteristic is a quantity indicating the achievement or the result of a data transmission operation, the *transmission rate* (R) or the amount of information transmitted per unit time, usually expressed in bits per second (R_B). Some publications prefer the natural logarithm or the decimal logarithm in place of the logarithm of base two. Correspondingly, the rate may also be expressed in nits per second or in dits per second. In systems with time-variable rate, an *average transmission rate* ($\overline{R}$) must be defined and the averaging interval should be indicated.

The second characteristic is a quantity indicating the quality of the achievement. This is usually the *error ratio* (e), i.e., the ratio of the number of digits received in error to the total number of digits received. This ratio should be determined at the output of the system. It is advisable to express this output error ratio as a ratio of the number of bits in error to the total number of bits delivered to the user, although some authors prefer to define the output error as the ratio of the number of bits in error to the number of bits correctly received by the user. Both definitions of the *bit error ratio* (e_B) asymptotically approach each other when the number of errors approaches zero. However, it is very important to notice that many systems are delivering the output information to the user in nonbinary form. An example is the characters of the alphabet printed by a teletypewriter. In such cases it is necessary to measure the *character error ratio* (e_c) and to calculate the *equivalent bit error ratio* (e_{EB}). The conversion equation between these two magnitudes is to be determined in line with the design of the decoder. If no information about the nature of the character error is delivered to the user, the equivalent bit error ratio will be highest. If full information can be forwarded to the user, such as indicating the second and third most likely received character along with *the* most likely received character, the equivalent bit error ratio will be smallest.

The upper and lower bounds for the equivalent bit error ratio when the character error ratio is measured are as follows:

$$e_c/n < e_{EB} \leqslant e_c/2 \quad , \tag{2}$$

where n is the number of bits carried by one character. If all m_c characters of an alphabet are used equally frequently, equation 3 applies:

$$n = \log_2 m_c \tag{3}$$

The reader is referred to the following publications for additional information: Okunev (64), Servinskiy (64), Beinhocker (61), Cohn (63), Filipowsky and Muehldorf (65a, Appendix B), Silverman and Balser (54), Hackett (63), Posner (64).

The third and fourth characteristics are two quantities which indirectly determine the dollar cost of the system and its equipment. They are the bandwidth of the transmission channel and the signal power of the transmitter at the input to the channel. The *bandwidth* (B) is in some cases completely at the discretion of the designer (certain space channels); in other cases, it is an absolutely fixed constant given to the designer (telephone channels). The actually available bandwidth rarely can be completely defined by a single number (in Hz). In some cases it is defined by regulatory bodies (Federal Communications Commission in the U.S.A.); in other cases, by the characteristics of the transmission medium and amplifying equipment (frequency response characteristics of telephone circuits); and, in still other cases, by the characteristics of the receiving equipment (filters in phase-lock loops, Karras 65). This situation forces designers to arrive at an *effective bandwidth* (B_{eff}) when attempting to compare nonbinary systems which operate over channels of different bandwidth. The effective bandwidth can be easily derived from a frequency response curve of the channel (Karras 65).

It is also important to recognize that many systems perform bandwidth translations through which they either spread out the energy over a much wider band (band spread system) or concentrate it into a much smaller band, when going from the input signals (base-band) to the transmission signals (carrier band). In these cases one must discriminate between the bandwidth of the information-carrying inp

signals at the transmitter (B_{in}), the bandwidth of the transmission channel (B_{tr}) and the bandwidth of the information-carrying output signals (B_{out}) (Filipowsky and Muehldorf 65a, section 4.1.4). For the purpose of this paper, we assume that the input information and the output information are in binary form, and we are concerned only with the transmission bandwidth B_{tr}. The notation B will always be understood as the effective bandwidth in the transmission channel unless stated differently.

The *signal power* S is the second parameter which is partially at the discretion of the designer. Naturally, there are constraints to an unlimited increase of the signal power.

One such constraint in telephone circuits is the maximum voltage or current for which the components and lines are rated. Another constraint is the danger of cross talk between circuits if the power that is used in some of the links is too large. Particularly severe limitations apply to circuits going over carrier systems because these systems are designed with carrier-suppressed speech circuits in mind (Thiess 60). This means that the frequency divided (FD) multiplex equipment is loaded only during periods of active speech and, of course, only a small percentage of the total time used for a telephone conversation is filled with active speech (Holbrook and Dickson 39). If many circuits of an FD multiplex system are simultaneously loaded with data channels, an overload situation may result which will give less power to the individual circuits than assumed in the design calculations and which may also cause undue distortions.

In radio circuits the power constraint is in the capability of the output amplifier devices, which are either average power limited or peak power limited. Specially designed high-pulse transmitters can achieve particularly large peak power to average power ratios and thus are particularly suitable for certain pulsive systems.

The notation, S, will designate the average power of the information-carrying signals when integrated over a duration of many signal intervals. For many calculations it is actually more important to use the signal energy in place of the signal power. It is defined as :

$$E_s = \int_{t_1}^{t_1} f_s^2(t)\,dt = S_s\,.\,(t_2 - t_1) \tag{4}$$

E is the average value over a large number of energy expressions of signal symbols or signal characters. We particularly use the expression E_D to define the average signal energy of one digit ; i.e., of one symbol out of the alphabet of m different symbols. In many cases all symbols of the alphabet will have the same energy, but there are exceptions, such as multilevel systems. A most important magnitude is E_B , the signal energy per bit of output information. Considering the definition of S and of R, one arrives at the relationship :

$$E_B = \frac{S}{R} = S\,.\,T_B \tag{5}$$

in watt-seconds, with T_B being defined as the bit period in seconds.

The unique magnitude for the characterization of white noise is the *noise power density* (N_o), which is defined as the average noise power (N) per unit bandwidth :*

$$N_o = \frac{N}{B_{tr}} \tag{6}$$

in watt-seconds.

*" White " noise is defined as noise with a uniform power spectrum, i.e., with N_o = const. (independent of frequency). In some theoretical investigations the power spectrum is plotted for positive and negative frequencies. We refer N_o to the one-sided power spectrum, avoiding a factor 2 in the denominator of equation 6.

Noise is produced at many places in the channel and in the receiver. The most sensitive spot where noise can enter a data link is a place which is followed by high gain amplification, i.e., the input to the final amplifier or radio receiver. It is normally this place to which all other noise sources are referred and their total action is then expressed as the average operational noise temperature ($\bar{T}_{op}$) (Filipowsky and Muehldorf 65a, pp. 176-247). If this magnitude is supplied, then N_o can be calculated from :

$$N_o = K_{BO}\,.\,\bar{T}_{op} \tag{7}$$

in watt-seconds. K_{BO} is the Boltzman's constant 1.38044 $\times 10^{-23}$ VAS/°K.

These simple relationships of equations 6 and 7 are not sufficient to describe more complex noise models. At this time, however, practically all mathematical models of nonbinary systems have been developed with white Gaussian noise as the only disturbance. Though the author is completely aware of the insufficient approximation to real life channels, which these models offer, he is also convinced that the white Gaussian noise model is an interesting and mathematically easily accessible basis for a first comparison of the many nonbinary modes of operation. It will, therefore, be used as the basis of the trade-off charts to be discussed in the next section. Whenever appropriate, comments will be inserted to indicate the possibilities, and also the weaknesses of any comparison of actual field tests with these highly idealized mathematical models. Ten to twenty years from now, engineers will have highly improved mathematical models of special channels ; performance predictions will be possible with adequate accuracy ; and computer simulations will permit fast experimentation within wide ranges of parameters, without ever building any of the systems under consideration. But the students will still have to use the simplified models which we are using here, before they will be able to understand the highly sophisticated more accurate models.

Basing the present considerations on the white noise model with Gaussian amplitude distribution, we first notice how important it is to measure or to specify how much the signal stands out of the noise. The quantity historically used for this purpose is the signal-to-noise power ratio (SNR), i.e., the power advantage which the signal can claim above the noise. This term, also called the power contrast, will be symbolized by the Greek letter σ :

$$\sigma = \frac{S}{N} = \frac{S}{N_o\,B_{tr}} = \frac{S}{K_{BO}\,\bar{T}_{op}\,B_{tr}} \tag{8}$$

in numbers.

There are many different locations in a receiver where the power contrast can be measured. Section C3 will present a short review of the important signal processing characteristic, which is nothing else but the graphical presentation of the output SNR (σ_{out}) as a function of the input SNR (σ_{in}) of a nonlinear signal processing device (Filipowsky and Muehldorf 65a, pp. 176-247).

Closely related to the SNR is a magnitude (ρ) defined as the average signal power over the noise power density :

$$\rho = \frac{S}{N_o} = \frac{S\,.\,B_{tr}}{N} = \sigma\,.\,B_{tr} \tag{9}$$

in Hz.

This quantity has the dimension of a frequency (S^{-1}) and it is something of a *characteristic rate* of the system, also called " signal contrast frequency " (Filipowsky and Muehldorf 65a, pp. 176-247). Equation 9 shows that it is the product of the transmission bandwidth and the power contrast. One can see that a system with a power contrast of 10 (10 db) has a characteristic rate of 10 times the trans-

mission bandwidth. From this, one concludes that the ratio between the actually achieved rate and the characteristic rate will be something like a measure of the communications efficiency of a system.

The difficulty when operating with ρ, the characteristic rate, is the fact that it is not a normalized magnitude like σ, which is a pure number and which may be conveniently expressed in decibels, a logarithmic measure. Many authors used, therefore, another dimensionless magnitude which has been called the *energy contrast*, or E over N_o (Van Horn 61, Muehldorf et al 59). We shall use the Greek letter ϵ for this important quantity, which, under the previously mentioned restraints, is defined by :

$$\epsilon = \frac{E}{N_o} = \frac{S \cdot T}{N_o} = \sigma \cdot B_{tr} \cdot T \tag{10}$$

in numbers.

The energy contrast can be defined for one digit (ϵ_D, E_D, T_D), for one bit (ϵ_B, E_B, T_B), or for any other convenient signal energy and time unit. Notice that N_o also has the dimension of energy (power over bandwidth). The bit energy contrast ϵ_B is, therefore, a ratio of the energy which the designer is willing to spend for transmitting one bit of information to the energy of the adverse effects (N_o). It will be shown in section C2 that the ideal communications system postulated by Shannon as the upper bound of the channel capacity needs exactly one unit of ϵ_B to transmit without errors one bit of information per second in a channel of 1 Hz bandwidth.

The sixth and last quantity of importance for the evaluation of nonbinary data system is a *measure of the distortions* which the information-carrying signals will suffer between transmitter and receiver. In section B3 a large variety of distortions was listed which may affect the various transmission media. Apparently no single number can be found which could serve as an adequate measure for characterizing all these effects. For the purpose of this paper we shall assume that all distortions may be negligible (linear transmission channel).

C2. *The Utility Chart*

The six basic characteristics, discussed above, will now be used in a practical design chart.

(1) Transmission rate (R_B)
(2) Error ratio (e)
(3) Transmission bandwidth (B_{tr})
(4) Signal power (S)
(5) Noise power (N)
(6) Distortion measures

No practical chart should deal with six variables at the same time. Design engineers need normalized measures; i.e., a reduction of the number of variables by defining relative magnitudes. They also prefer dimensionless quantities, such as efficiency, density, or other "percentage" measures. Many attempts have been made in the past to arrive at such design charts for various modulation methods (Jelonck 52, Helstrom 60, Sanders 59b, 60a). The chart used in all the following sections is a modification of several of these earlier attempts. It has demonstrated its practical value in our laboratories. The following logical considerations lead automatically to this chart.

The *first step* is to eliminate all second-order considerations. Thus distortion measures are not considered during the first approach. We assume linear channels. Distortion measures can always be introduced as fixed parameters.

The *second step* is to simplify any quantities that cannot be expressed in simple terms, by making restricting assumptions. This has been indicated in section B1 by using N_o as a single number representing the most regular kind of noise and by introducing an effective transmission bandwidth, B_{tr}.

The *third step* is to identify magnitudes which serve merely as limiting bounds and plot any charts for fixed parameter values of such magnitudes. This can be done with the error ratio. Most of our curves have been calculated for a fixed bit error ratio of 10^{-3} (i.e., for one bit in error out of 1,000 bits of information received at the output of a link).

After these simplifying steps, one must still deal with four important quantities : R, B_{tr}, S, and N.

The *fourth step* is to normalize by using ratios of these magnitudes. These ratios should be dimensionless. There are only two physical dimensions attached to these four magnitudes. R_B and B_{tr} have the dimension sec^{-1} and S and N have the dimension watts. Thus we define a new magnitude :

$$D_B = R_B/B_{tr} \tag{11}$$

in numbers.

This magnitude may be called the *information density* or, when R_B is expressed in bits per second, the *bit density*. It is defined as the rate in bits per second over the transmission bandwidth in cycles per second. For example, a bit density of one bit per second per Hz (i.e., a bit density of one) will permit a system to send 3000 bits per second over a channel of 3000 Hz. It is convenient to express D_B in decibels following the equation :

$$D_B^{db} = 10[\log_{10} R_B - \log_{10} B_{tr}] \tag{12}$$

in decibels.

In this logarithmic measure, negative db numbers will indicate a rate less than the bandwidth and a positive number will indicate a rate higher than the bandwidth. If the rate in bits per second is equal to the bandwidth in Hz, then the bit density is zero decibel.

A ratio of the other two remaining magnitudes, S and N, has already been defined in equation 8 as the power contrast (σ, power SNR). It would, therefore, be convenient to plot σ as a function of D or vice versa, and to declare that system the better one which requires the smaller power contrast for achieving the same bit density or which achieves a higher bit density for the same power contrast. Indeed, this normalization has been used many times in the past for comparing binary systems with sampled or analog communications systems. Nonbinary systems, however, may be used with equal advantage in ranges of very small power contrast and also of very high power contrast. In both these cases they will operate at about the same energy contrast. It is, therefore more advantageous to use a presentation which plots the bit density as a function of the energy contrast. A chart of this nature was first introduced by Sanders (60a). The version introduced here has an additional advantage over Sanders' chart in that both axes increase in the normal way from left to right and from bottom to top when the corresponding magnitudes improve in the desirable direction.

We define, similarly as Sanders, a kind of communications efficiency. We call it the *utility* (u) of the system. It is defined as the ratio of the achievement (the bit density D) to the effort to reach this achievement (the power contrast σ) : For the ideal system the utility assumes the value 1 when the signal power and noise power are equal. If a practical system needs a signal power larger than noise power, it has a utility smaller than one. Thus the utility may be expressed in percentage values, quite similar to the efficiency in mechanical engineering. The utility is related to the other magnitudes in the following way :

$$u = \frac{D_B}{\sigma} = \frac{R_B}{B_{TR}} \cdot \frac{N}{S} = R_B \cdot \frac{N_o}{S} = \frac{R_B}{\rho} \tag{13}$$

$$u = \frac{D_B}{\sigma} = N_o \cdot \frac{1}{T_B \cdot S} = \frac{N_o}{E_B} = \frac{1}{\epsilon_B} \tag{14}$$

$$u = \frac{D_B}{\sigma} = \frac{R_B}{B_{TR}} \cdot \frac{1}{\sigma} = \frac{1}{T_B \cdot B_{TR}} \cdot \frac{1}{\sigma} = \frac{1}{\beta_B \cdot \sigma} \qquad (15)$$

in numbers.

Equations 13, 14, 15 show many ways of relating the utility to the fundamental magnitudes defined in section C1. Equation 13 shows that the utility is also the ratio of the actually achieved transmission rate R_B to the characteristic rate of the system. Equation 14 shows the very important fact that the utility is nothing but the reciprocal value of the energy contrast. Equation 18 shows that the utility can be kept constant when trading off signal power against the so-called time-bandwidth product, $T_B\,B_{TR}$. For simpler notation the letter β is introduced to characterize the time-bandwidth product, frequently also called the *signal base*.

$$\beta = T \cdot B_{TR} \qquad (16)$$

The signal base may be taken for a bit of transmitted information (β_B), for a digit (β_D), or a whole message (β_M). In each case one must select the correct value for the signal duration (T_B, T_D, T_M). The signal base, like most of the other characteristic magnitudes, is dimensionless.

The importance of choosing this utility as the key figure-of-merit of data systems can still be better recognized when plotting the utility of Shannon's ideal and hypothetical error-free communications system in a logarithmic scale in decibels. This is done in fig. 1. Shannon's upper bound of the channel capacity of this ideal system for an average power limited transmitter is usually written in the following form (Shannon 48) :

$$C = B \cdot T \cdot \log_2\left(1 + \frac{S}{N}\right) \quad . \qquad (17)$$

C is the maximum number of bits that the ideal system can transmit in the time (T) without error. Introducing the notations of this paper :

$$R_B = \frac{C}{T} = B_{TR} \log_2 (1 + \sigma) \qquad (18)$$

$$\frac{R_B}{B_{TR}} = D_B = \log_2 (1 + \sigma)$$

$$\sigma = 2^{D_B} - 1$$

$$u = \frac{D_B}{\sigma} = \frac{D_B}{2^{D_B} - 1} \qquad (19)$$

$$10 \log_{10} u = 10 \log_{10} D_B - 10 \log_{10} (2^{D_B} - 1) \qquad (20)$$

$$u^{db} = D^{db}_B - 10 \log_{10} (2^{D_B} - 1) \qquad (21)$$

$$u^{db} = D^{db}_B - 3.01 \cdot D_B \qquad \text{for } D_B > 10 \qquad (22)$$

$$u^{db} = -10 \log_{10} [0.6931 + 0.2404 \cdot D_B + 0.0555 \cdot D_B^2 + \ldots] \qquad \text{for } D_B < 0.1 \qquad (23)$$

Equation 21 is the equation for the utility curve of the ideal system, which is plotted in fig. 1 as curve 1. Its upper side is shaded to indicate that this curve represents an upper bound for all other systems under the stated assumptions of white Gaussian noise and average signal power limitations. The interesting fact is that this system leaves complete freedom to operate at large bit densities (right side of fig. 1) or small bit densities (left side). It also shows that the ideal system has the utility 1 (0 db) for a bit density of 1 (0 db).

There is an approximate expression for small values of D_B (equation 23), which indicates that the utility for the ideal system can exceed the value 1, but that the highest utility ever to be reached is $-10 \log_{10} 0.6931 = 1.592$ db. This value has been frequently quoted in the literature as $(\log_e 2)^{-1} = 1.441$ in numbers.

Another approximation, given in equation 22, is valid for bit densities above 10. Notice that D_B in the first term is expressed in decibels but, in the second term, it is expressed in numbers.

Bit density = rate of xmission / Bandwidth

Noise Power Density
Power Density per Bit

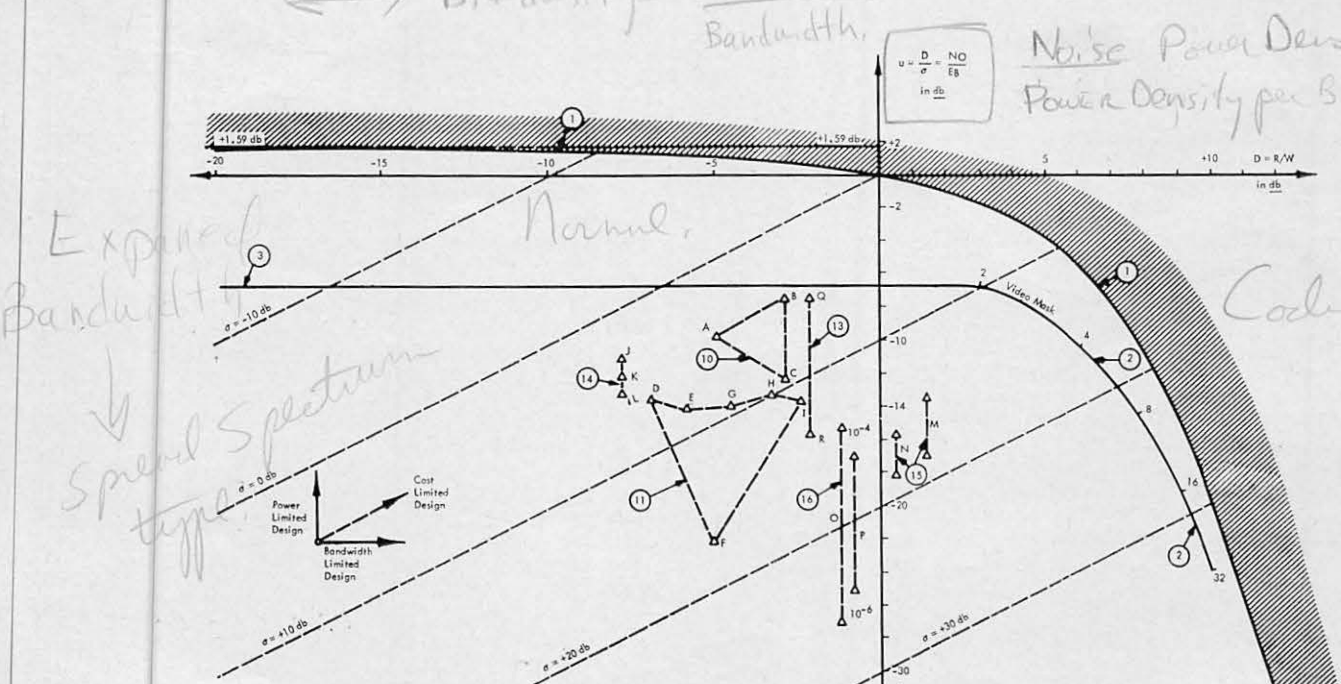


Figure 1.—The utility chart for the ideal communications system (Shannon 1948) and for a number of practical data systems.

Figure 1 shows, in addition to the curves for the ideal systems, the location of a number of practical systems in the same chart. This should demonstrate the value of the chart as a means for comparing the achievement of practical systems with their theoretical possibilities, but it should also indicate the large range for improvement that is still available. Most practical systems operate 10 to 20 db below their theoretical utility ; i.e., they operate with less than 10 percent efficiency. The dashed diagonal lines in fig. 1 represent lines of constant power contrast. The line for zero db power contrast goes through the origin of the chart where it meets the ideal system.

The best way to understand the value of the utility chart is to start with a hypothetical design at the most inefficient location (the left lower corner) and to attempt an improvement in efficiency. The goal is the ideal system, line 1. There are three fundamentally different approaches.

(1) *The power-limited design* : One tries to increase the utility by reducing the power which the transmitter has to supply, while keeping the transmission rate and bandwidth constant (vertical line going upwards).

(2) *The bandwidth-limited design* : Here the opposite is the case. The bandwidth is prescribed (telephone channel) but the power may be increased or the noise may be reduced by design improvements. The principal direction of the design improvement is to the right.

(3) *The cost-limited design* may be followed when both bandwidth and power are at the discretion of the designer or if he is limited in both values to some extent. This design follows in general a constant σ line.

The individual lines and points in fig. 1, which are marked by capital letters, represent the practical systems discussed below.

(1) *Operating points ABC* : A teletypewriter system tested in the United Kingdom in laboratory tests with Gaussian noise generated by a noise generator (Rosie 63). The purpose of these tests was to demonstrate the advantages of higher order detection of complete teletypewriter characters as compared with a bit-by-bit detection of the individual elements of a character.

Point B is the theoretical value for an error ratio of 10^{-3}. This point shows how well a design could operate in the best case. It would use an alphabet of 32 waveforms, each consisting of 5 binary elements (20 ms long). This theoretical model assumes that a correlation detector and maximum likelihood decider (MLD) will be applied.

Point C is the operating point for a standard bit-by-bit teletypewriter receiver. It loses utility of approximately 5 db due to the sub-optimal detection method of point C. This point results from measurements in the laboratory when adjusting the power contrast to a value that keeps the error ratio below 10^{-3}.

Point A is representative of measurements on an experimental model with correlation detection which had only 10 characters (the 10 figures). This reduced the transmission rate, and thus the bit density, to 0.37 or -4.4 db. Considering the circuit deficiency (filter slopes) brings a small increase in the transmission bandwidth, giving a final value of -5.0 db for the bit density of the experimental model. It is interesting to note that points A and B are almost on the same line of constant power contrast ($+5$ db). This example shows how the utility chart can be used to compare experimental designs with theoretical models. It likewise demonstrates the value of the chart when comparing systems with different transmission rates.

(2) *Operating points DEFGHI* : This area is representative for laboratory tests of U.S. Army data transmission equipment. Special test equipment was designed for this program to generate data, detect errors, and record the error statistics in a useful manner (Tucker and Duffy 63). Various modems (data transmitter and receivers) were tested while operating over carrier transmission equipment with white noise as disturbance.

Points DEF are for binary FSK transmission. They are included as a comparison and also to show the influence of distortions even for such simple waveforms. The carrier equipment was an AN/TCC-7 12-channel set for field cables. Point D represents the results after passing one channel ; point E is reached after passing through 3 channels in tandem ; and point F, after passing 12 channels in tandem.

Points GHI are the results when sending the same messages over a four-phase modem at 1200 bits per second.

(3) *Operating points JKL* are examples of operational results to be discussed as a demonstration of the advantages of the utility chart. The data is extracted from a recent tabulation of the operational communications results of the U.S. deep space flights of Pioneer and Mariner (Reiff 66). The derivation of the bit density and the utility from this publication presents some difficulty as the paper does not give full information about the bandwidth of the data channels. The difficulties in deriving the utility result because a part of the transmitter power goes into the synchronization channel and the ranging channel. A term called " modulation loss " gives the signal power in db which has to be deducted for these purposes. Considering this loss and using the signal and noise levels specified in the paper, one arrives at a utility of -11 db for Pioneer V and Pioneer VI (point J), of -12 db for Mariner II (point K), and -13 db for Mariner IV (point L).

This example may show that it is possible to place operating points for rather complex systems on the utility chart. In this case we learn that the space data systems are the systems with the lowest bit density of all the practical systems that are considered in this set of examples. This was to be expected, because such systems operate with very small power contrast (SNR). Evidently, this is the top requirement for space-to-earth links. Indeed, the point J has the lowest power contrast ($+3$ db) of all examples in this set. On the other hand, the Pioneer/Mariner examples may serve as an introduction to the complicated problems of deriving bit density and utility from practical measurements on operational systems. A more detailed analysis of the actual equipment would possibly place these systems more to the left.

(4) *Operating lines MN* : Line M evidences the performance of the most frequently applied nonbinary modulation method—the four-phase digital data transmission, also called the quaternary phase keying mode. In particular, the lines are derived from a test report of the Hughes Aircraft Company (HC 270) modem, a digital data receiver (Evans et al 61). Line N indicates results of a vestigial sideband (VSB) binary modem as a comparison. Lines M and N apply to field tests over leased telephone lines (length estimated by this author at 250 miles). The lengths of the lines M and N indicate the margin which should be reserved when different kinds of leased lines would be used. The VSB system (line N) required carefully equalized line facilities while the nonbinary system (line M) was able to operate over nonequalized facilities. This is the reason that the margin for N is smaller : anticipating that the equalization procedure will make different kinds of lines nearly equal. The VSB system (N) needed 2160 Hz bandwidth to operate at 2400 bps

while the 4-phase system (M) needed only 1800 Hz to operate at 2500 bps. This is why line M is placed approximately 1 db to the right of line N.

This example clearly demonstrates the advantages of a nonbinary system—higher bit density and higher utility. More will be said in section H about MPSK systems.

(5) *Operating lines OP* : These are results from tests performed in Germany, also with 4-phase data transmission equipment, but with pulsive disturbances intentionally inserted (Von Hänisch and Kettel 63). The line O is for a coherent detection system ; line P is for a noncoherent detection method. The upper ends of both lines are the points for bit error ratio of 10^{-4} ; the lower ends are the points for a bit error ratio of 10^{-6}.

These German tests indicate the rather disturbing influence of pulsive noise on data transmission systems, particularly when a very low error ratio is desired. The utility is brought down to -26 db.

(6) *Operating line QR* : This last example has been selected as an interesting case introducing a novel fading model and demonstrating the theoretical performance of certain nonbinary systems over channels following the characteristics of this model. The model is called the BFC, the binary fading channel (Motley and Melvin 66).

This example is intended to show that we can compare, in the same utility chart, the performance of systems operating over linear channels and others operating over nonlinear channels, provided that a mathematical model is defined for the nonlinear channel (BFC, in this case).

In summarizing the common aspects of all the six examples, it can be stressed that the utility chart emerges not only as a practical tool for the comparison of binary and nonbinary modulation methods, but also for the comparison between any digital systems with different bandwidths, different rates, or different coding procedures. SNR, bandwidth requirement, transmission rate, and error ratio can be readily read off from the utility chart. More about the utility chart and its applications may be found in Filipowsky 67c, 67d, 68d.

C3. *Threshold Systems*

The last of the six characteristics which are used to describe nonlinear data systems was introduced as "distortion measures". This was supported by a statement that, for the purpose of this paper, linear operation of the channels will be assumed and distortions measures will be introduced as a second-order correction. We now amplify this statement to include two purposely introduced, severe nonlinearities which do not violate the assumption of a basically linear channel. They will be summarized under this heading of threshold systems.

The first nonlinearity is the *threshold in the decision system*. At the end of the linear transmission channel, there must be a device that restores the original digital character of the messages. This device, usually called the decision system, is characterized by the error probability function, which will be discussed later. Sometimes decision systems are used in the transmitter for converting analog information into digital form. This A/D conversion process causes the so-called quantization noise (Crow 62). In this paper we are not concerned with quantization noise, nor with the closely related distortions and noise due to the sampling process (Stewart 56). In all cases we assume digital input and digital output for the nonbinary systems.

The second purposely introduced nonlinearity is the *improvement threshold in FM systems*. Many nonbinary systems will operate over basically linear channels which go over receivers with completely nonlinear IF amplifiers (containing chains of limiters). Still we have to consider these facilities as linear channels as far as the information-carrying signals are concerned. However, these channels must not be treated as Gaussian channels as far as the noise is concerned. The limiters change the statistical characteristics of the noise. The utility chart can be used in such cases by combining it with the signal processing characteristic of the nonlinear receiving system (Filipowsky and Muehldorf 65a, Section 4.1.4). A recent paper (Klapper 66) provides a link between the threshold mechanism in analog FM systems and the error ratios in digital FM reception. An earlier paper (Von Baeyer 63) considered the same problem, including the effect of intersymbol distortions due to band limitations. Two other investigations may be mentioned as examples of the many studies that have been devoted to FM threshold problems (62, Campbell and Herbert 65). Finally, it should be pointed out that phase lock loop circuits show also a typical improvement threshold, whose influence on digital transmission signals may be of importance (Becker et al 65). Beyond these few references we must omit the very complex nonlinear problem from this paper except for the first nonlinear problem, the decision threshold.

To analyze the decision process, one can use the conventional error curves which are plotted in most publications and which are derived for a special threshold decider model. For example, fig. 2 shows on the right side the error curve for an ideal binary system, marked $n = 1$. This chart shows the utility vertically in decibels $u = 10 \log_{10} (N_o/E)$ and the bit-error ratio $\bar{e}_B$ horizontally. The other curves on the right side ($n = 2, 3, 4, 6, 8, 10$) are the error probability curves for the most efficient nonbinary system, Viterbi's coded phase-coherent communications system (Viterbi 61). The left side of fig. 2 is a utility chart as discussed in section C2. The lines A-E are the utility curves for an ideal binary system in DSB-SC (double sideband, suppressed carrier modulation). It is assumed that coherent detection can take place and that the signal extractors will be matched filters matched to two orthogonal waveforms. We also assume a zero-threshold decision device, which will have the error curve marked $n = 1$ at the right side. Curve A is then the utility curve for a bit-error ratio of 10^{-1} ; curve B, for 10^{-2} ; and so on until curve E, which is plotted for a constant bit-error ratio of 10^{-5}. One can now read off how many db utility one loses if the error ratio should be improved, say, from 10^{-3} to 10^{-4} (only about 1 db), while an improvement from 10^{-1} to 10^{-2} would cause a utility loss of 4.7 db. To find these curves when the error curve is given, one needs merely search for the point on the error curve which corresponds to the desired error ratio. This is the constant utility value for the respective utility curve. The particular nature of the ideal binary system assures the horizontal run of these curves.

To demonstrate how more general utility curves are derived from error curves, we use Viterbi's error curves. From Viterbi's paper one can derive the following relationship :

$$D_B = \frac{R_B}{B} = \frac{n}{2^{n-1}} \tag{24}$$

$$D_B^{db} = 10 \log_{10} n - 3.01 (n - 1) \tag{24a}$$

A coded phase-coherent communications system of $n = 6$ has accordingly $M = 64$ different orthogonal sequences and operates with a bit density of -7.27 db. The utility for 10^{-3} error ratio can be gained from the error curve marked $n = 6$. It is -4 db. This would give one point of the utility curve for 10^{-3}. However, there is one practical aspect which should be considered. Equations 24 have been derived by Viterbi under the assumption that the system

operates with minimum bandwidth. He considered only one-half the elementary rate as the transmission bandwidth. This assumption is justifiable if the system operates in a wide band channel with no stringent bandlimitations. However, if the system should operate over a bandlimited channel or if the transmitter is restrained from radiating any frequency components outside the nominal transmission bandwidth, distortions will be unavoidable. Such distortions may indirectly cause additional errors and the system may require a much higher power (operate with lower utility) to obtain the specified low error rate. The optimum design will then be the one which avoids distortions by increasing the bandwidth when higher quality is needed. This is shown in fig. 2 by shifting the utility curves to the left when going to lower error ratio. Curves F to J show this family of utility curves. Because of this offset for a comfortable bandwidth margin, it happens that all points for a constant length of sequences (n = const.) lie on a slightly curved vertical line (dashed lines). Notice also that for n = 1 the points of the utility curves of a coded phase-coherent system are also the right endpoint of the ideal binary DSB-SC system. For n = 2 (quaternary Viterbi system), the bit density is the same as for n = 1, again a clear advantage for the nonbinary system. Notice also that the utility is truly improving for all curves with increasing length of sequences. However, the longer the sequences, the smaller is the improvement for another doubling of the length of sequences. An optimum length will soon be reached when hardware costs have to be considered. More will be said about these systems in section K.

Decision devices may also have the capability to refuse a decision. This is normally done by giving the device two thresholds, declaring the zone between the two thresholds a null zone. If the observed magnitude at a decision instant has a value in the null zone, no decision will be made (Bloom et al 57).

C4. *Measurement of Performance*

Ultimately, it is the measurement of the performance of a system during operation that will decide its fate. Binary systems are well established: they are simple and their performance is good. If nonbinary systems do not demonstrate equal or superior performance, any cost, speed, or other advantage will not be adequate to bring the sweeping success that some people predict. The understanding of measurement and test procedures is, therefore, important for all equipment developers who hope for a strong market of their new systems.

The six characteristics developed in section C1 and used directly or indirectly in the utility chart are also the six most important magnitudes to be measured.

The *transmission rate* is easy to measure in systems with constant rate. They are normally clocked and the clocking rate is slaved to a highly accurate master clock. Operation is impossible unless transmitter and receiver are synchronized. Thus the receiver follows with the same precision that the transmitter has in generating messages. A simple checking of the total transmission time with a stop watch and the automatic counting of the total number of transmitted message elements will establish the average rate. Systems with variable or adaptive rate present only a slightly more complex problem, but averaging of the message count over shorter intervals (seconds) in a running registration of these counts will provide an "instantaneous" rate sufficiently accurate.

The *error ratio* is more difficult to observe. The usual practice is to loop the test link back to its origin, if this is a feasible approach (Bartelink and Knight 64). Transmitter and receiver can then be located next to each other, and input messages and output messages can be directly compared in coincidence circuits, provided the former ones will be delayed for an amount equal to the total loop transmission delay. This procedure also gives, as a by-product, this total delay time. Noncoincidence of message elements will be counted as errors. Precaution must be taken that no systematic noncoincidence error might be established within the test equipment.

Interesting test facilities have been developed for the statistical recording and evaluation of errors. Bason (66) describes equipment for the experimental evaluation of a system with multilevel pulses (MASK) and for the counting of its errors. The Hughes Aircraft Company designed a special digital error measuring set (DEMS) (60a) and Enticknap (61) discusses the equipment used by the Lincoln Laboratory of MIT for data transmission tests over telephone circuits. The equipment is called ADDER, acronym for Automatic Digital Data Error Recorder (Hofmann 61). Maholick (65) reports on project COMFACTS, which involved a specially designed central processor similar to the IBM 1401. The processor provides specifications to describe the performance and reliability of any data transmission installation under test. Small (65) applied a technique that provides

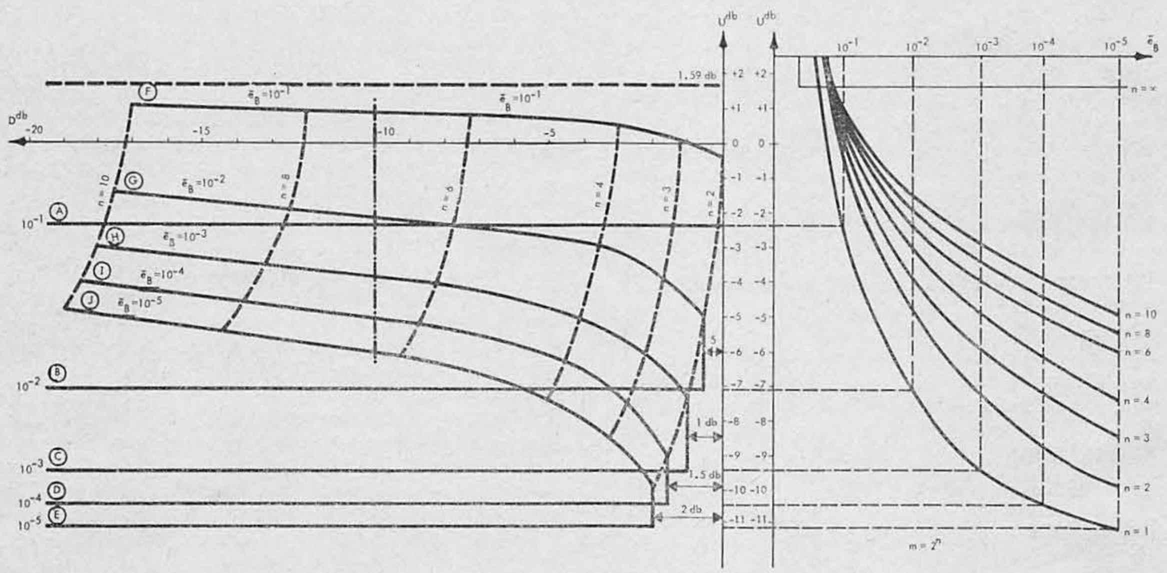


Figure 2.—Generating utility curves (left) from the error probability curves (right).

time compression in the recording of error patterns. This permits longer automatic recording periods as well as reduced testing and analysis costs.

The *effective transmission bandwidth* is in many cases a standard value prescribed by the public carrier or any national or international communications administration. In other cases it must be derived from frequency response measurements with the help of a simple graphical procedure. If its measurement is needed, the methods are the same as for any other frequency response measurement.

The measurement of the *signal power* would be a simple procedure with standard instruments except for the fact that it must be determined at a rather low power level, where usually noise is present. Thus one actually measures the power of the mixture of signal and noise. If it is also essential to know the power distribution (i.e., the power spectrum of the signals), one will normally measure the total spectrum of typical messages with a signal analyzer and determine from this curve the signal power within the transmission band. Similarly, it is possible to find the peak power from oscillographic observations of the time functions of typical message signals or from direct measurements with the PAR (peak to average power) meter (Anderson 64). Attenuators in laboratory tests or attenuation measurements in field tests must then establish the power level at various places in the system.

The *noise power* can be evaluated when the signal power is temporarily switched off the system, provided the channel is perfectly linear. If automatic volume controls, limiters, and other nonlinearities are involved, one may err greatly when concluding that the noise power without the signal present would be the same as the noise power during the correct operation of the system.

The same difficulties apply to the direct measurement of the SNR (Levitan and Peysikhman 64). The SNR monitoring of deep space facilities presents particular difficulties because of the extremely weak signals reaching the antenna. The Deep Space Instrumentation Facility in the U.S.A. has analyzed two automatic monitoring methods of high precision (Gilchriest 64) which may be of interest for application in other areas.

Signal energy or energy contrast are rarely measured directly. The duration of signals or messages is normally well known and the energy can be calculated when the average signal power is known.

Particular problems are created by pulsive noise (Bodonyi 61; Fennick 65) and by time-varying channels (Bukhviner 64; Sharkey et al 65; Korte 65; 65a). In the first case, one usually resorts to laboratory tests with pulsive noise recorded on magnetic tapes as it has been observed over the circuits to be used (Fennick 65). In the second case, special measurement and recording methods are applied to explore the statistical behaviour of the medium. Once this is known, simulation methods are applied in the laboratory to duplicate this behaviour under well-controlled conditions. In both cases, the experimenter should operate the different test objects (competitive system) under realistic but practically identical test conditions.

Simulation is also applied to compare various error-correcting methods under realistic pulsive noise conditions (Bennett and Froehlich 61). Eye patterns are a relatively new method of a more subjective inspection of transmission channels and modulation methods with the help of oscillographic devices (Davey 64, Von Hänisch and Kettel 63).

Complete test installations for the evaluation of data systems over telephone and telegraph circuits are described by Purtov et al (65) for testing Russian facilities, by Ebenau and Stotesbury (64) for testing West European facilities, by Tucker and Duffy (63) and Bartow (65) for testing U.S.A. line modems for Army applications, and by Lamb (63) for testing U.S.A. Army mobile tactical equipment. A Russian paper reports a method for a statistical study of telegraph characteristics using a digital computer (Tyakhti 65). Ansorge (65) describes a general test set for the evaluation of the quality of a voice-frequency data transmission set and staff members of the Jet Propulsion Laboratory (JPL) investigated the characteristics of test messages for supervising their world-wide surface communications network on Earth (65b).

The above-mentioned publications are only a small fraction of all the reports, papers, and descriptions of tests and test facilities for communications plants. Those selected are either directly concerned with testing of nonbinary systems or they develop the material which we think is of importance for this purpose.

D. Mathematical Models

Mathematical models are indispensable for the analysis of any electronic system. For nonbinary systems they actually are the point of origin. Many new ideas have evolved from mathematical models, and many more papers have been written about the mathematical models than about the practical application of nonbinary systems.

A mathematical model can best be understood when explained in terms of a block diagram. Figure 3 is an attempt to show the functional units of a very general nonbinary system operating over a single channel. Let us explain this diagram in connection with an octonary MFSK system. This is a system where eight different frequencies are available to carry the information.

Figure 3a shows the five key elements in any nonbinary system.

Figure 4 shows the sequences of operation in the particular MFSK system. The encode block separates nine binary input elements into three trigrams A, B, C and forwards these to the signal formation block. Following the signalling alphabet on the top of fig. 4, the signal formation device will create frequencies f_3, f_4, and f_6, which go in this sequence over the channel where the signals are contaminated. The extraction device on the receiving side will give an output signal for each of the possible eight frequencies. After the first signal A″ arrives, assume that there is only one output voltage (of peak value) at the output for f_3. All other outputs are zero. This corresponds to a noiseless and distortion-free transmission. Afer the second signal B″ arrives, there are several output voltages, but the highest is still the one of frequency f_4 which was actually transmitted. After the third signal C″ arrives, there are two outputs higher than the others—f_2 (highest) and f_6 (second). As we know that f_6 has been transmitted, we may anticipate that the receiver will make an error.

Indeed, if the following decision and decoding device follows the so-called maximum likelihood rule, it will, for each decision, compare all the eight outputs, determine the largest one, and conclude that this is the signal that has actually been transmitted. However, the information might have been encoded in more complex form, permitting the decider to perform more complex calculations which, under certain circumstances, can tolerate large errors or erasures in some of the signals of a large group and still arrive at a correct final decision.

In the foregoing example, we assume that the decider is bound to make an error when interpreting the last digit and will deliver to the output the group $++-$ (f_2) in place of the correct group $-+-$ (f_6). Notice that an error in the interpretation of an octonary digit may cause either one, two, or three binary output errors. If the equipment also considers the runners-up (i.e., the outputs with the second and third highest output), it frequently can assure that single binary errors result from a transmission error and that multiple errors are avoided. This is why it is impossible to

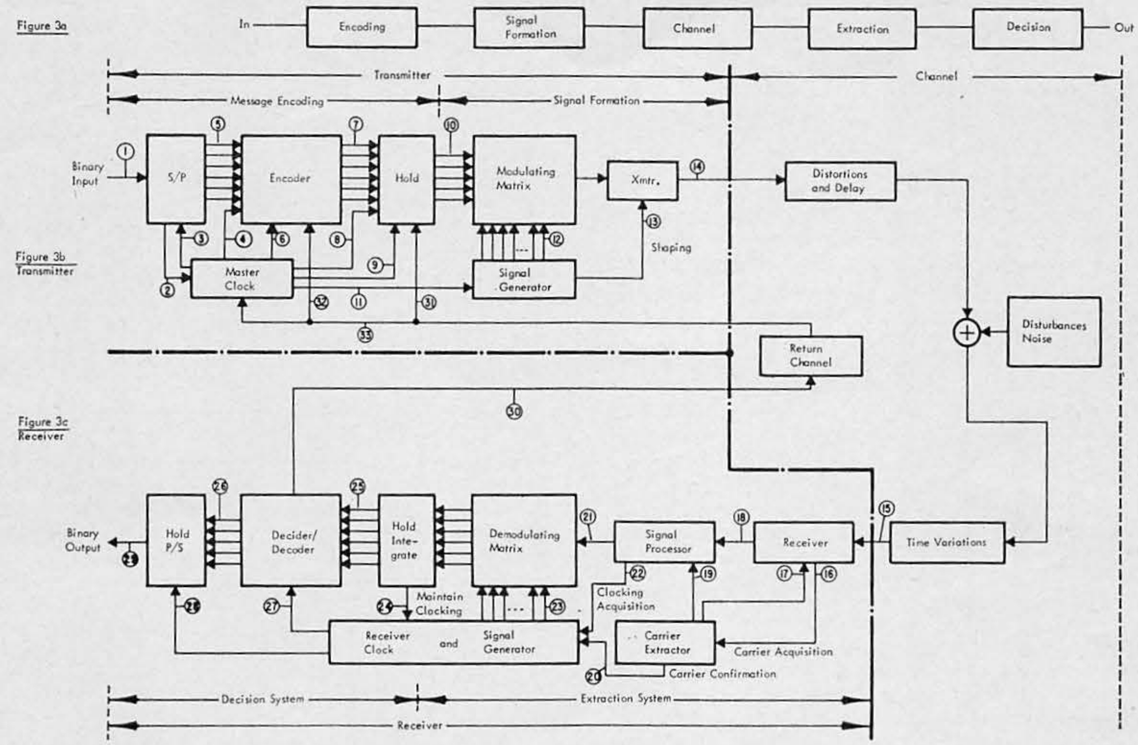


Figure 3.—Block diagram of a general nonbinary transmission link.

ENCODING PROCESS

Input	1.	+	+	+	+	-	-	-	-
	2.	+	+	-	-	+	+	-	-
	3.	+	-	+	-	+	-	+	-
Transmit		f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8

DECISION PROCESS

Largest Output at		f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
Decide	1.	+	+	+	+	-	-	-	-
	2.	+	+	-	-	+	+	-	-
	3.	+	-	+	-	+	-	+	-

EXAMPLE

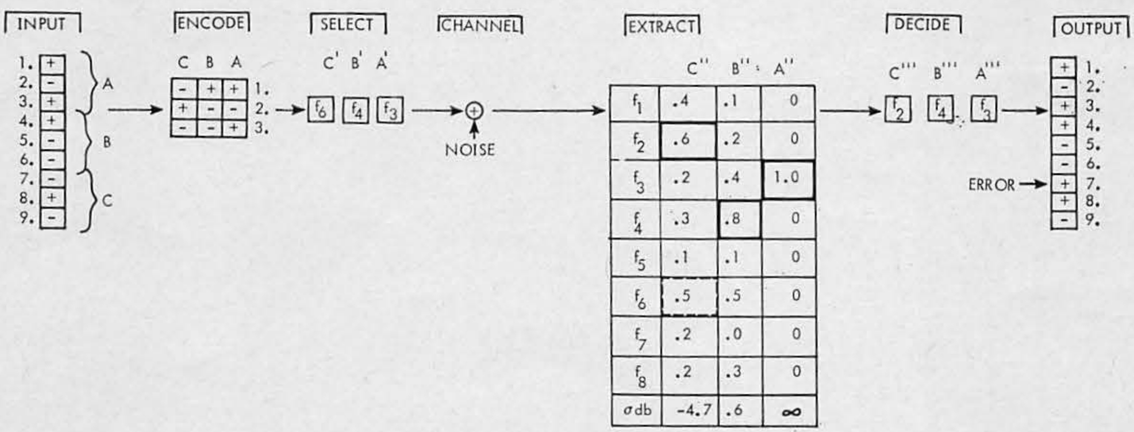


Figure 4.—An example of the nonbinary transmission of a 9-bit message over an octonary MFSK system.

establish a fixed relationship between transmission character error ratio and output bit error ratio without knowing the decoding procedure. Upper and lower limits for this relationship are quoted in equation 2.

Assuming that the eight transmission signals are orthogonal to each other, the approximate power contrast can be determined by dividing the square of the highest output by the sum of the squares of all lower outputs. The results are marked as σ db in fig. 4.

Returning to fig. 3, part b (transmitter) and part c (receiver), one sees the essential functional elements within each of the five units of fig. 3a. The encoding unit consists of a serial to parallel converter (S/P), the actual encoder, a block of hold circuits, and the master clock. The latter can be slaved to the input rate via line 2 ; it will, in turn, control the rates of all operations of the encoding subsystem. The clock drives the signal generator via line 11 and secures the transmission of synchronization information over the channel to the receiver. This is symbolized by connection 8.

The signal formation subsystem consists basically of the signal generator and the modulating matrix. Examples will be given in the following sections of the design of these units for the various nonbinary systems. In the example of the octonary MFSK, the signal generator produces the eight frequencies at parallel outputs, feeding them over eight connections (12) to the modulation matrix. This device consists merely of eight switches, each of which passes the corresponding frequency to the transmitter (Xmtr) when triggered from the hold block. Connection 10 likewise consists of eight lines. If the output bandwidth is to be strictly limited, a shaping pulse may be sent over line 13 to suppress the output amplitude smoothly at the time of the sudden transition of one frequency to the other.

The transmission channel is symbolized by a block representing signal delay and signal distortions, a linear adder superposing the disturbances (noise) and the signal, and, finally, another block representing the time variations (multiplicative disturbances).

The receiving terminal consists of an extraction section and a decision section. The former, as the name implies, extracts the class of desired signals from any other signals which may reach the receiver input. This is normally done by selective amplifier circuits, which pass only the desired transmission band. Low noise performance of this most sensitive part of a link is essential. In particularly sensitive installations, there are special circuits for the extraction of the transmission carrier. In extreme cases, one must go through special carrier acquisition procedures (16) before the carrier extractor can be activated. The carrier, once restored in the receiver, can be used to improve the signal extraction process. This is indicated by connection 19, which leads to the signal processor. This term is used for widely differing devices. We restrict its meaning to incorporate all nonlinear analog operations on the transmission signals. These are particularly important whenever the information is carried in angular modulation. They are normally known as limiters, discriminators, signal or frequency tracking devices, etc. (See section C3.) Sometimes it is difficult to specify exactly which circuits of a receiving installation should be included into a particular one of the three units : carrier-band receiver, signal processor, and carrier extractor.

The demodulating matrix is usually, but not always, the counterpart to the modulating matrix. Most systems operate with the one or the other kind of a synchronous (coherent) demodulator. For this operation, noise-free and distortion-free replicas of the transmitted signals are generated locally in the receiver. They are then multiplied with the noisy received signals and, in many cases, the products are integrated and stored in the " hold " circuits. To perform this operation correctly, the signal generator must be phased exactly to the incoming signals. This is done via connection 22 ; the operation is normally called *clocking acquisition* or *epoch acquisition* or simply *synchronization*. Once the demodulator matrix is in perfect operation, the signal generator can be locked in by a closed-loop control operation to maintain perfect clocking even under time variations (Doppler) of the transmission channels (24).

The decision part of the receiver consists of the decider/decoder unit, a hold circuit block, and a parallel to serial converter (P/S) unit. All these units must be slaved to the receiver clock via connections 27 and 28.

The receiver operation in the octonary MFSK system can be simplified. In the extreme case one would retain only the demodulating matrix, the decider, and the output circuitry. The former would consist of eight bandpass filters, each tuned to one of the eight frequencies. Each of these would be followed by a full-wave envelope detector with a time constant approximately equal to the duration of one elementary octonary signal. The decider could be self-synchronized to the peaks of the demodulator output. Beyond this oversimplified receiver arrangement there are many more complex, but also more efficient, versions, using all the other special circuits displayed in fig. 3.

With this general block diagram of a nonbinary transmission system (fig. 3), one can now look for mathematical models describing the key units. Notice that several magnitudes in these models will be random processes. There are many good textbooks available on statistical mathematics and on the fundamentals of random processes. Particularly suitable for the requirements of communications systems are a recent Russian review (Tikhonov 64), a German review (Von Szalay 64), and an unclassified U.S. Government report (Goodman 64).

The most general mathematical model of a nonbinary system results from C. E. Shannon's historically famous paper (Shannon 49). It derives the upper bound of the bit density of an ideal not further specified communications system with the help of multidimensional geometrical mapping considerations. Many of the more specific mathematical models to be discussed in the next sections go back to this fundamental paper. A more recent paper should be of highest interest to analytical workers as it attempts the partial ordering of nonbinary channels (Chang, T. T., and Lawton 62). This paper contains further references to other fundamental papers on this critical problem which have been published by C. E. Shannon and other in the 14 years between these two publications. J. K. Wolf (62) points out that a clarification of terminology and an agreement on several fundamental magnitudes in noise theory is essential when comparing mathematical models of different authors.

Floyd and Nuttall (65) published in May 1965 a U.S. Government report which is available to the public. They prove that nonbinary systems are always superior to binary systems even if the latter ones use error-correcting coding procedures. In this report is a rather elaborate mathematical model which applies four different levels of nonbinary coding procedures.

All the investigations mentioned so far present mathematical models of the most general nature. They do not specify special waveforms, nor special modulation methods or circuits. This will be done in the remaining Parts of this paper.

The staff of JPL has an interesting answer for those who should doubt the usefulness of mathematical models in general. They compared Viterbi's model of an orthogonal coded phase-coherent nonbinary system with the results of precision measurements and found the agreement to be very satisfactory (63a).

D1. *Transmitter Models*

The transmitter model specifies the nonbinary system and prescribes coding procedures and signal waveforms. Modulation methods are usually included in the specification of the signal waveforms. When specifying the transmitter model, one must know the predictable channel characteristics, and the transmitter model should match the channel characteristics in the most efficient way. Usually there are several receiver models that can be used with a given transmitter and a given channel model. One of these will be optimum in a theoretical sense, although a different one may be the most efficient in an economical sense. Utility curves will present the total efficiency of any selected combination of the three kinds of models.

Shannon (49) hinted at a transmitter model in connection with a block diagram, similar to the one given in fig. 3, by discriminating between the message space and the signal space. The coding subsystem of fig. 3 can be taken as the device forming the most suitable message space from the original information flow. The signal formation subsystem

can be taken as the device forming Shannon's signal space. For the latter one Shannon states that " It can be shown that (its) coordinates are all perpendicular to each other and are obtained by what is essentially a rotation of the original coordinate systems." As his original coordinate system, Shannon used the system of sin x/x functions equally spaced $\tau = 1/2B$ seconds apart. His foresight clearly indicates that the mathematically convenient sin x/x system will not be the ideal coordinate system for practical cases; rather, he suggested a rotation of the coordinate system to arrive at a more practical signal space.

What Shannon could not disclose in 1949 was the mechanism of arriving at those desirable characteristics that could bring a practical system close to the ideal systems. More detailed mathematical models had to show the path. A refinement was needed for answering a question in connection with Shannon's effect number 5, the processing delay: How close can one approach the ideal system when tolerating a specified finite delay? Mathematically, this answer required the solution of the " sphere-packing problem ", which was geometrically introduced by Gilbert (52) in his search for optimum signal alphabets in the sense of Shannon's 1949 recommendations. The sphere-packing problem received more complete attention by Balakrishnan (61) and Weber (65). Technically, the easier accessible approach to optimum systems with finite delay seemed to be the use of the more familiar coding techniques with binary elements.

Again it was Shannon (57) who attempted in mathematical studies to arrive at a model of an ideal system under the restraint of a finite coding interval. His 1959 paper, enhanced by Slepian's practical interpretation (Slepian 63), provides such a model. Figure 5 is a representation of Slepian's charts in the units used for the utility plot of this paper. Slepian uses six magnitudes, similar to the six magnitudes of section C1 in the present paper. He performed a computer evaluation of the improvement which an ideal nonbinary system could reach over a binary system. He assumed a gradually increasing encoding delay which would not reach the infinite delay postulated by Shannon in 1949 as the fifth characteristic of the ideal system.

Shannon (59) and Slepian 63) use the maximum number of dimensions of the signal space as specified by Shannon in 1949 and earlier by Nyquist in 1928, i.e., the so-called Nyquist rate $2\beta_W = 2BT_W$. They define " block codes " of length l_0 equal to the number of dimensions of a signal. Section K will give full details of the Shannon/Slepian results.

We see in fig. 5 that the limitation of the encoding interval to five dimensions forces a designer to be satisfied with a utility of at least 5 db below the ideal system when his requirement is to have less errors than 1 in 10,000. For an error ratio of 10^{-6}, he pays a penalty of at least 6.4 db below the ideal system. It is also interesting to know that the smallest penalties apply to systems with 3 db and 5 db bit density. Systems with lower or higher bit density require a still higher penalty. One can approach the ideal system closer when extending the length of the block codes to the $l_0 = 25$ dimensions and still closer for $l = 101$. These l_0 values happen to be the values for which Slepian (63) has reached a numerical solution of Shannon's (59) equations.

The historial efforts to follow the guidance of the theory included such primitive modes as passive ternary systems (Filipowsky 58) and proceeded through the interesting ideas of case coding (Bason 64) to the linear real coding approach of Pierce (66) and the permutation modulation of Slepian (65a) to be discussed in Part III.

The general concept of signal space as a multidimensional space has received much attention, not only for the purpose of nonbinary communications systems. A few of the many papers in this special mathematical discipline may be of particular interest. Kirshner (62) described the interrelation-

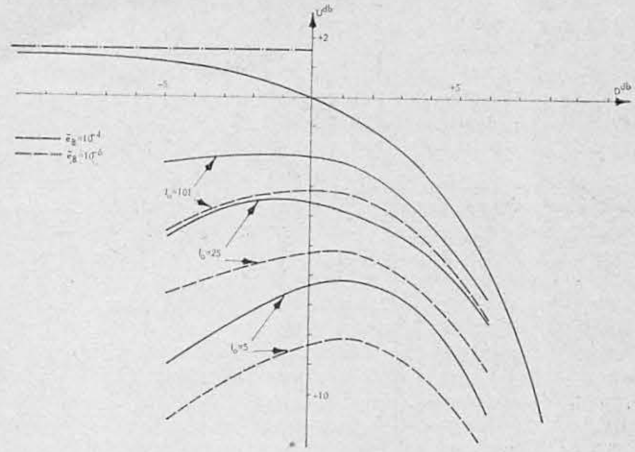


Figure 5.—Utility curves of an ideal nonbinary system with 5, 25 or 101 dimensional signals and error ratios of 10^{-4} and 10^{-6}.

ship between signal space, modulation, and bandwidth in a rather practical manner which may appeal to many engineers. Bello (64) developed a concept of duality, called time-frequency duality, which is applicable to a class of networks called communications-signal-processing networks. He illustrated the benefit of this approach by constructing the dual of the Kineplex communications system. (See section H.) M. Zakai (65) investigated the problem of bandlimited functions and bandlimited (but not necessarily stationary) processes in relation to the sampling theorem. There are also interesting recent contributions by Kailath (65), Huang and Gabrielian (66), and R. S. Bennett (66).

Turning now from the mathematical models of the message encoding susbystem (fig. 3) to those of the signal formation subsystem, one must first recognize the basic problems of generating *one-dimensional bandlimited signals*. Most systems arrange these signals in time sequence, which raises the problem of intersymbol interference. We briefly referred to this problem in section B1. The mathematical model of the signal formation specifies the kind of signals which cause the minimum intersymbol interference and still permit a high signal density. Practical signal design will remain, to some degree, an art, and adjustments will have to be made to match the signal formation subsystem to a given channel. Signal space diagrams in the form of Lissajoux figures or eye patterns will be helpful in such matching efforts (Davey 64).

When generating mathematical models of one-dimensional signals, one should know the characteristics of so-called *elementary signals* and of their signal bases (TB products). Gabor (46) used Gaussian-shaped signals, and Ville (48), Oswald (56) and Silverman (58) investigated *analytical signals*. These are mathematical models of elementary signals which use a complex time domain and a complex frequency domain. In this way it is possible to treat pairs of related signals (Hilbert transforms) in one and the same analytical operation. Tikhonov and Levikov (65) compare the β-values (TB products) of a number of pulsed signals including rectangular signals, Gaussian-shaped signals, and transients of devices with from one to five resonant circuits. Other contributions to the analysis of the shape of elementary signals are available from Chalk (49), Muehldorf et al (59b), Gurevich (60), and Kennedy (60). Giovannoni et al (65) described a technique for synthesizing bandpass waveforms by modulating a sampling waveform at several instants, i.e., actually generating a sequence of sampling waveforms. Weaver (62) published an analysis of the practical aspects of signal formation for various pulse modulation systems and concluded that a signal base of 5 is required when intersymbol interference should remain below −30 db. Levi (65) solved

the problem of fitting an arbitrarily bandlimited signal to a finite number of arbitrary points in the amplitude-time domain with a signal of minimum energy. An interesting signal with dual properties is described by Lender (65a). These signals are elementary signals, which also can be classified as multidimensional signals. Part IV will give more details about practical transmission signals.

Mathematical models of signal formation subsystems for *multidimensional signals* are more essential than those for one-dimensional bandlimited signals. We stressed at the beginning of this section that a system approaching the ideal system requires signals of very high dimensionality. The designer of such signals now has full control over the intersymbol interference between the components of the signal. The only remaining interference out of his control is the one close to transition points between adjacent multidimensional signals. Thus we see that the importance of intersymbol interference reduces with increasing dimensionality of the signals. The practical engineer who works with " long " signals is not much concerned with their " fringe effects ". On the other hand, signal distortions may lead to misinterpretations of some of the signals of the resulting very large alphabets. The complexity of such large sets of signals makes it imperative to use well-defined mathematical models.

The decreasing influence of neighbouring signals permits the application of a rule of thumb, linking the dimensionality of a signal to the signal base that is required to transmit the signal with reasonably small distortions :

$$\beta = \mathrm{TB} = \frac{l^2}{2l - 1} \tag{25}$$

l is the dimensionality of the signal or, in other words, the number of orthogonal coordinates which can completely describe a signal of the signal base β. According to this rule, a one-dimensional signal ($l = 1$) requires a signal base (TB product) of 1, while a signal of large dimensionality needs only a signal base of 1/2, i.e., of half the dimensionality.

Most mathematical models of multidimensional signals follow Shannon's example and describe the signals as vectors in a multidimensional space. Various *orthogonal coordinate systems* may be used and, indeed, one arrives at an orthogonal set of signals, when placing all signal points on the orthogonal axes of such a coordinate system. (Orthogonal signal sets are discussed in greater detail in Parts III and IV.)

Arthurs and Dym (62) explained the application of orthogonal coordinates with the help of an example in two-dimensional space. Their general mathematical model is of great importance for the theory of all multistate systems. Lachs (63) showed how to maximize the minimum distance for a set of m points on an n-dimensional sphere, a problem which has to be solved when designing suborthogonal digital systems.

Many investigators concentrate on particular orthogonal functions as the basis of a coordinate system for multidimensional signals. Recently, strong attention has been given to systems of prolate spheroidal wavefunctions as orthogonal coordinate systems. (Details will be discussed in section N4.) Once an orthogonal coordinate system is established, interesting rotational transformations can be performed on the complete set of signals to match the signal characteristics better to the channel characteristics (Lang 63). Nuttall and Amoroso (65) applied orthogonal coordinate systems to find a set of m orthogonal signals with minimum Gabor bandwidth.

D2. *Channel Models*

Channel disturbances and channel distortions are the basic limitations to higher data rates. Nonbinary systems will be able to fulfil their promises only when their designers have the tools to describe the channel characteristics with the needed precision. In sections B2 and B3 we listed all the adverse effects which designers have to consider. Here we review the large effort, going on at present and still increasing, which ultimately will provide convenient mathematical models for all practical channels.

Again it should be stressed that the model of a linear distortion-free channel disturbed only by additive white Gaussian noise is very useful as a reference standard. As a first approximation, it is certainly true that a system which excels against this ideal channel model will also perform well against other, more realistic, channel models. For further refinement, a system should be modified so that it is best matched to the specific channel characteristics at hand. The ideal model is adequately covered in many textbooks, although the treatment of special problems like the " axis crossing " is more likely to be found in journals (Helstrom 57, McFadden 56 and 58).

Elliott's (65) paper on a *model of the switched telephone network* for data communications may show us how well the " model builders " are progressing. This example includes a short historical review of channel models and demonstrates how the application of realistic channel models simplifies the analysis of error control systems and the determination of error rates for error control codes. The paper also indicates the complexity of such models by using about one dozen parameters and arranging them into a mixture of three distinct channels. Computer simulation will be unavoidable when applying such models on a large scale.

Time varying channel characteristics are not included in Elliott's model, but they are very important for radio links. Bello and Nedlin (62a, b ; 63, 64) analyzed the performance of selectively fading channels when used for binary data transmission. Turin (58) established a model for channels with nonselective slow fading and referred to still older contributions to models of general multipath channels as they result from scatter propagation modes. Reiger (57) considered still earlier a model of a fading channel. One Russian paper discussed the performance of a reference signal for clocking data transmission receivers when subjected to fading (Gerastovskiy 65) ; another Russian paper used the ratio of the static to the " scattering " power as the parameter to indicate the degree of fading (Servinskiy 64). A third Russian paper by Khvorostenko (64) investigated the performance of MPSK systems over channels with Rayleigh fading, one of the most frequently used models of fading channels. Reports of U.S. Government Laboratories compared the theoretical models with the practical results (Greim et al 65, Korte 65). This leads to the definition of a " fading bandwidth ", a term which may find more application in the future.

Special disturbances have been investigated extensively. The most *general models* are those which consider non-stationary noise components. Belousov (65) investigated noise characterized by the product of a Gaussian stationary process with a slow-varying function of time. Other attempts to generalize the mathematical models of external disturbances, in contrast to the more regular internal receiver noise, led to models which assume that the noise characteristics are only partly known or that the noise is correlated to the signals (Nesteruk and Porfir'yeva 65, Musa 63).

Impulse noise, as indicated in section B2, is the most critical disturbance in telephone circuits. Numerous investigations in the last ten years provided the basis for a number of convenient mathematical models which describe approximately the characteristics of this kind of noise. Mertz (65), in a recent report published by the Rand Corporation, summarized the results as they were available early in 1965. From this report and from other original contributions one learns that Gilbert (60) presented a simple on-off model in the form of a Markov chain, with one state representing

error-free transmission and another state representing a transmission period with constant error ratio. In 1960 Mertz conceived a more sophisticated model that incorporated a bi-modal model for the burst duration. One mode consisted simply of 1-bit bursts; the other mode was represented by a continuous triangular distribution of burst durations. This model was published in a Rand report in 1960, but the report was released in 1965. Berger and Mandelbrot (63) demonstrated, with the aid of experimental data obtained from the German telephone network, that the distribution of inter-error intervals can be well approximated by a law of Pareto of exponent less than one. Comparison with experiments led Sussman (63) to conclude that the Pareto model provides an excellent representation of the error statistics. More recently Engel (65) used a model consisting of bursts of the carrier frequency with random phase, occurring randomly in time. This rather simple structure of the model permitted Engel to analyze various modulation systems in their performance under such noise environment. Lindenlaub (65) claimed that a still simpler model consisting of rectangular single pulses of constant area shows good agreement with experiments.

Models of *atmospheric noise* are not yet as numerous as those of impulsive noise in telephone circuits. Reinmann (65) used a distribution for atmospheric noise which was originally derived in a paper by Beckmann (64). Reinmann's thesis contains 19 references to related papers. A Japanese paper investigated the crossing-rate distribution of atmospheric noise envelopes (Nakai 65), and Bello (65b) combined the previously discussed fading model with a model of atmospheric noise, thus arriving at a realistic model for the comparison with actual digital communications systems operating over HF links.

Errors may result even in noise-free transmission circuits. Hill (62) established mathematical models which permit the comparison of the influence of distortions with the influence of noise. Lukatela (64) showed, in a German paper, that similar models may be applied to account for the influence of delay distortions.

This review may convince the reader that communications engineers are at the beginning of an area where even seemingly unpredictable events such as atmospherics, fading, and other distortions will find their correct place in mathematical models. We quoted only a small selection of all the publications in the area of communications media and their characteristics. We attempted to select only those references which have the strongest bearing on the problems of nonbinary systems.

D3. *Receiver Models*

The receiver models represent the lower part of fig. 3 marked 3c. Mathematical models describing the performance of a data receiver may range from general models, postulating basic reception theories, to specific models, describing individual circuits with all their shortcomings. Good examples for the first class are the three original contributions by Van Meter and Middleton (54) and Middleton and Van Meter (55a, b), laying the foundations of modern decision theory for communications systems. These very general models, which incorporate all classes of binary and nonbinary systems, permit the derivation of general rules of optimization. These models cannot be used directly to predict the performance of any given implementation or circuit. For such performance predictions, models of the second class are needed. A good example is a paper by Karshin (64) describing a mathematical model of a synchronous digital receiver (in this case, for binary signals). This model permits the comparison of many circuit variations and can be used to predict the detailed performance of any one of them.

Apparently both classes of models are needed. The class of *general models* has progressed to some degree of specialization. Wyner's theory of bounded discrepancy decoding (Wyner 65), for example, can be applied to many classes of channels. The paper itself studies the theoretical capabilities of minimum distance decoding (MDD) and bounded distance decoding (BDD) for four different classes of communications channels, including binary, nonbinary, and continuously sampled channels. This theory soon may lead to improved decision subsystems. Kadota (65) reports a theory of the optimum reception of m-ary Gaussian signals in Gaussian noise. Although the title suggests a further specialization, the theory is virtually a generalization by including probabilistic signals (following a Gaussian distribution similar to that of the noise) while most of the other theories apply to " sure " signals. Kadota (64) earlier developed such a generalization for binary Gaussian signals. That paper contains many references to the earlier literature about Gaussian signals (also called statistical signals, probabilistic signals, stochastic signals, etc.). The decision process deals here with the discrimination or the measurement of statistical parameters, such as covariance, moments, and others. Theories of this nature are most important in connection with unpredictably time-varying channels. In such channels, not only the noise is unpredictable but the signals also become partly unpredictable. A typical channel with such characteristics is the moon bounce relay. Kasowski (63) described a nonbinary system operating with noise-like signals over such a time-varying channel.

The *extractor models* are either based on cross-correlation extractors (Fano 51) or on matched filter extractor (Reiger 53, Turin 60). The former variety, indicated in fig. 3c, is more flexible. The mathematical model of a cross-correlation extractor has been investigated by many authors (George 54, Lange 63, Haas 61, Wierwillie 65, Ludovici 63). Most of them point out that it needs still further improvement, primarily when applied to multidimensional signals. Investigations of the output of multipliers when two or more random variables operate at the input (Lampard 56a, Donahue 64, Springer and Thompson 64, Lezin 65a, b, Cooper 65) may be helpful in this direction. Watanabe (60) performed an information theoretical analysis of multivariate correlation which, though it aims at different applications, may be of importance to nonbinary signal extracting systems.

The *decider models* have been described in general terms by the early contributors to decision theory (Kotel'nikov 59a). Reiger (58) was possibly the first to apply the maximum likelihood decider (MLD) to a practical model of a nonbinary system (MFSK). A good review of the decision subsystem for equicorrelated m-ary signals, together with many references to earlier contributions, is in Nuttall's (62) paper. Becker et al (65) compared the MLD with analog demodulation methods. Other contributions to the theory of decision subsystems are available in papers by Rudnick (61), Blasbalg (61), Muroga et al (61), and by Kohmenyuk (63). Two interesting " letters to the editor " are concerned with very special problems of the " largest-of selection systems " (Cooper 63) and the Bayes decision procedure (Ward 62). Hackett (63) derived an accurate estimate of the correlation performance of any group code perturbed by white Gaussian noise and compares it with the performance of digital decoding and direct transmissions of the same information. He concludes that correlation detection offers no more than 3-db power contrast advantage over digital decoding. This paper assumes that in each case a MLD is applied and that no further attempt is made to evaluate any runner-up decisions information. Indeed, only character error probabilities were evaluated by Hackett.

The *decoder models* are usually treated in connection with receivers for coded binary transmissions. Only two papers may be singled out for references in connection with decision models. One is Wyner's (65) bounded discrepancy decoding, already mentioned in connection with the discussion of

general receiver models. The other is a recent Master's thesis (Wakeley, Jr. 65). This thesis forms an interesting bridge between the general statistical sampling theory and the sequential decoding theory of communications systems. The latter theory is conventionally applied in connection with binary decisions and will, therefore, not be discussed specifically in this paper. Wakeley's thesis is much broader in its potential applications to nonbinary systems and may be of great interest to some research workers on nonbinary decision systems.

Beyond these mathematical models of extractors and decoders, we should like to mention a number of receiver models for special applications which do not lend themselves easily to a split into these two kinds of subsystems.

Polarity coincidence receiver models are much in use for radio astronomy receivers (Ekre 63). The *extraction of the epoch* of overlapping signals was the subject of a paper by Young (65). The *extraction of signals from a channel with correlated noise*, called " contrast reception " has already been mentioned under noise models (Nesteruk and Porfir'yeva 65). Daly and Rushforth (65) discussed the *nonparametric detection of a signal of known form in additive noise*, a method which may gain high importance for the detection of nonbinary signals in channels with non-Gaussian noise. Still more general may be the receiver model of Taylor (66) which operates with a nonlinear time-invariant filter for the *detection of signal pulses of known shape imbedded in nonstationary noise*. The problem of *distribution-free statistical detection* is discussed in a very general form by Bell (64) in a U.S. Government report which is available to the public. These methods of detection can be applied whenever the underlying signal and noise distributions are unknown. Although these ideas originated in the area of RADAR and SONAR detection, they may have a deep impact as detectors of nonbinary signals in an unpredictable noise environment. *Adaptive receivers* are another method of dealing with unpredictable channel characteristics. Many publications have been issued in the general area of adaptive communications systems. We restrict our review to the paper of Levin and Bonch-Bruyevich (65) which discusses a self-optimizing decision system and to the paper of Birdsall and Roberts (65) on an optimum nonsequential observation-decision procedure. Both papers seem to open some new avenues for the receiving methods in nonbinary systems.

Summarizing, we may point to the large variety of receiver models which are available for nonbinary receiving systems. It will be shown in the next Parts how a number of these models are already in use for the analysis of special nonbinary systems.

Part II—Multistate Systems

Introduction

Part II will discuss *basic approaches* for denser information packing. Graphical and numerical methods developed in Part I will be used to compare these various approaches. A basic block diagram of a general nonbinary system (fig. 3) will serve as the common link when explaining the differences between the various methods. All nonbinary systems presented in this part have one characteristic in common: the nonbinary signals, whatever their shape and mode of generation may be, are transmitted *one after the other*, in time sequence. The receiver, correspondingly, will make use of this advance information, and will identify in each transmission interval *one and only one* nonbinary message (character).

Sections F, G, and H discuss systems that respectively use the amplitude (MASK), the frequency (MFSK), or the phase (MPSK) of a sinusoidal carrier as the parameter that assumes one of the permissible states.

E. Fundamentals of Multistate Systems

The basic nonbinary information transmission systems are multistate systems, also called q-ary (or m-ary) systems. These terms express the fact that, during a given time interval called the message interval, the transmitter assumes *one and only one* out of q different states. All q states are known to the receiver in advance of any transmission. The most efficient reception method will be selected for any case at hand, making use of this *a priori* information. Many different methods are known for the specification of q states of a transmitter; the various sections of this part will describe the three basic q-ary nonbinary transmission methods.

The simplest waveform is one that keeps all its parameters constant over the whole transmission interval and then changes one of the parameters instantaneously to a new value when a new transmission interval begins which carries a new information character. This waveform is a rectangular pulse and, in its most general form, can be represented by a rectangular carrier pulse. Figure 6 shows that the pulse has six independent parameters, three of which are normally used (amplitude, carrier frequency, carrier phase) while a fourth, the epoch t_1 (i.e., the position of a signal within a message interval), is occasionally used in multiple time shift keying (MTSK) systems. The fifth parameter, the duration of a waveform, may likewise be used in MTSK systems, but will normally be fixed by the input rate and the number of subintervals in a message interval. The sixth, the dc component, could be used in certain channels to carry information, but in most situations it may suffer severe distortions or it may interfere with other service functions of the system, such as synchronization or Doppler compensation (carrier acquisition).

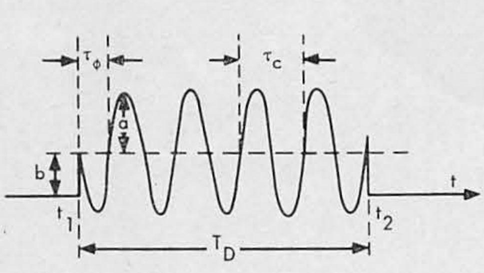
In the language of the block diagram of a general nonbinary system (fig. 3), we can specify for all systems of Part II the following set of conditions:

(i) The input will be binary.

(ii) The S/P translator will be necessary for all nonbinary systems.

The encoder will simply combine n binary input elements to one 2^n-ary internal trigger signal that will go to the " hold " circuit. This may go as a multilevel signal over a single line 7 or by a binary trigger signal over one out of 2^n lines (multiple frequency shift keying [MFSK]) or in any other convenient way. The modulating matrix and the signal generator may assume several forms, as will be discussed in the following sections. At the receiving side, there will be a similar variety of approaches for the arrangement of the carrier extractor, the signal generator, and the demodulating matrix. The hold and integrate circuit will be, in most systems, a single resampler, a comparator, or a summing network. The decider will be a simple single or multiple threshold device sending trigger signals to the " hold and P/S " device, which will finally restore the continuous binary serial output information flow. No direct influence of a return channel will be considered, although some of the practical systems plotted in the utility charts do employ communications feedback over a return channel.

Expressing the preceding assumptions in terms of fig. 4, it can be seen that these assumptions imply digit-by-digit transmission, the only mode considered in Part II. In fig. 4 the digit B′ transmitted by the choice of frequency f_4 will



$$f(t) = b + a \sin\left(\frac{2\pi}{\tau_c} t + \phi_c\right) \cdots t_1 \le t \le t_2$$

$$f(t) = 0 \cdots\cdots\cdots\cdots t < t_1 ;\ t > t_2$$

a · · · amplitude of carrier

b · · · dc component

t_1 · · · epoch, beginning of waveform

t_2 · · · end of waveform: $t_2 = t_1 + T_D$

τ_c · · · period of carrier: $1/\tau_c = f_c$ carrier frequency

ϕ_c · · · carrier phase: $\tau_\phi = \frac{\phi_c}{2\pi} \cdot \tau_c$

Figure 6.—The six parameters of a typical rectangular carrier signal.

be independent of the preceding digit A′ and the following digit C′. The extractor and the decider will consider the data for each digit separately and there will be no correlation of the extractor output over several digits. This restriction, together with the previously mentioned restriction that only one parameter will be shifted by the transmitter and that all other parameters will be held constant, characterizes all systems of Part II as one-dimensional binary or nonbinary systems. This statement holds, notwithstanding the fact that certain MPSK systems generate their signals in a two-dimensional way by superposing two phase-quadrature components. They still result in a one-dimensional transmission signal as long as they operate under the restraint of a constant amplitude. A truly two-dimensional generator must produce output signals with two degrees of freedom (in phase and in amplitude). Part III will include discussions of such multi-dimensional systems as well as serially encoded systems, which introduce purposely an interdependence of consecutive digits, even if all of them are carrying the information in the same and only parameter. As explained in Part I, in that case, each digit is a new dimension of the total message (or word).

Binary systems will be frequently considered, either as the basis of nonbinary systems or for the purpose of showing the improvement of the nonbinary system over the binary. In several cases it will be necessary to refer to systems which encode the information into several subchannels that carry their signals simultaneously (in frequency division) over the same channel (for example, Kineplex in section H). Yet, as explained before, the purpose of this paper does not include a complete survey of *polysignal* systems.

F. Multiamplitude Shift Keying (MASK) Systems

F1. *Ideal MASK Systems*

Binary systems inside computers and logical devices operate with rectangular signals, clearly the simplest signal waveform that anyone could use in multiamplitude shift keying (MASK) systems. Waveforms of this kind can be used in unipolar or bipolar mode. Figure 7a and b show examples of rectangular MASK waveforms with V_o being the peak-to-peak value in either case.

Assuming an ideal low-pass channel—a channel with a rectangular frequency response from dc to a definite cutoff frequency f_o and zero response for frequencies beyond f_o—leads to the sin x over x waveforms shown in fig. 7c. These waveforms, which were used in the early theory of MASK systems, were assumed when plotting the utility curve (No. 2) for an ideal low-pass MASK system (fig. 1). This curve derives from Oliver et al (48) ; also appearing as the upper bound for an ideal MASK system in fig. 8, it is drawn for a binary error ratio at the receiver output of 10^{-3}. This ideal MASK system may be considered a special case of the general nonbinary system of fig. 3, with the waveform alphabet consisting of q signals, all of equal shape, but with the amplitudes equally spaced between the negative peak value and the positive peak value (including the peak values). Figure 7d shows an example of a message formed by this bipolar system. The utility curve of an ideal system of this kind (No. 2, fig. 8) is based on the assumption of a linear, distortion-free low-pass channel with additive white Gaussian noise. The demodulating matrix of fig. 3 consists in this case simply of a precisely timed resampler, and the decoder is merely a digit-by-digit decider with thresholds exactly halfway between any two amplitude levels.

Another ideal system is presented by Arthurs and Dym (62) on the assumption that the unipolar (also called monopolar) rectangular waveforms of fig. 7a form the rectangular envelopes of a cosine carrier waveform of a frequency at the centre of the transmission band. The authors calculated the character error ratio of this system, assuming a linear distortion-free bandpass channel with white bandlimited Gaussian noise and either a coherent ideal matched filter extractor and demodulator or a noncoherent demodulator. The decider operates again with thresholds halfway between the equally spaced amplitude levels. We used Arthurs and Dym's curves and converted them to the magnitudes used in our utility chart (fig. 8a). For the coherent monopolar MASK system, this leads to the equation :

$$\bar{e}_B = \frac{1}{\log_2 q} \cdot Q\left\{\sqrt{\frac{3 \log_2 q}{u\,(q-1)\,(2q-1)}}\right\} \qquad (26)$$

In translating Arthurs and Dym's equation 59, notice that their error probability P_e is a character error probability. Assuming a low error ratio (single errors only) and assuming the application of a Gray code, we translated P_e into $\bar{e}_B$

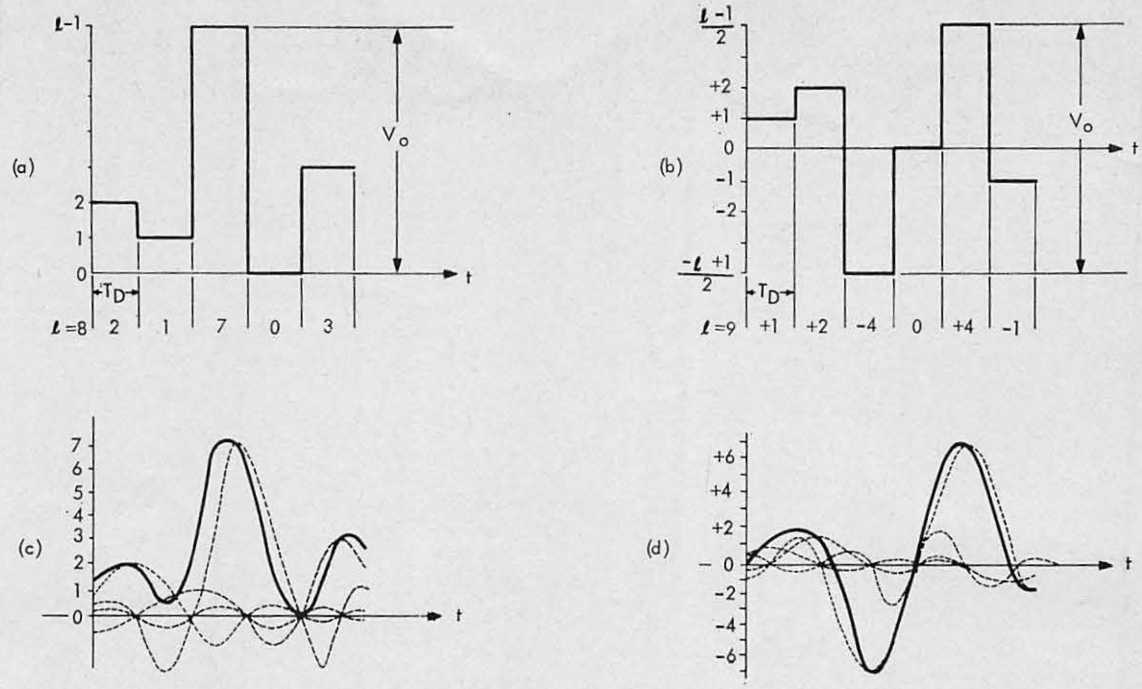


Figure 7.—Unipolar and bipolar low-pass signals.

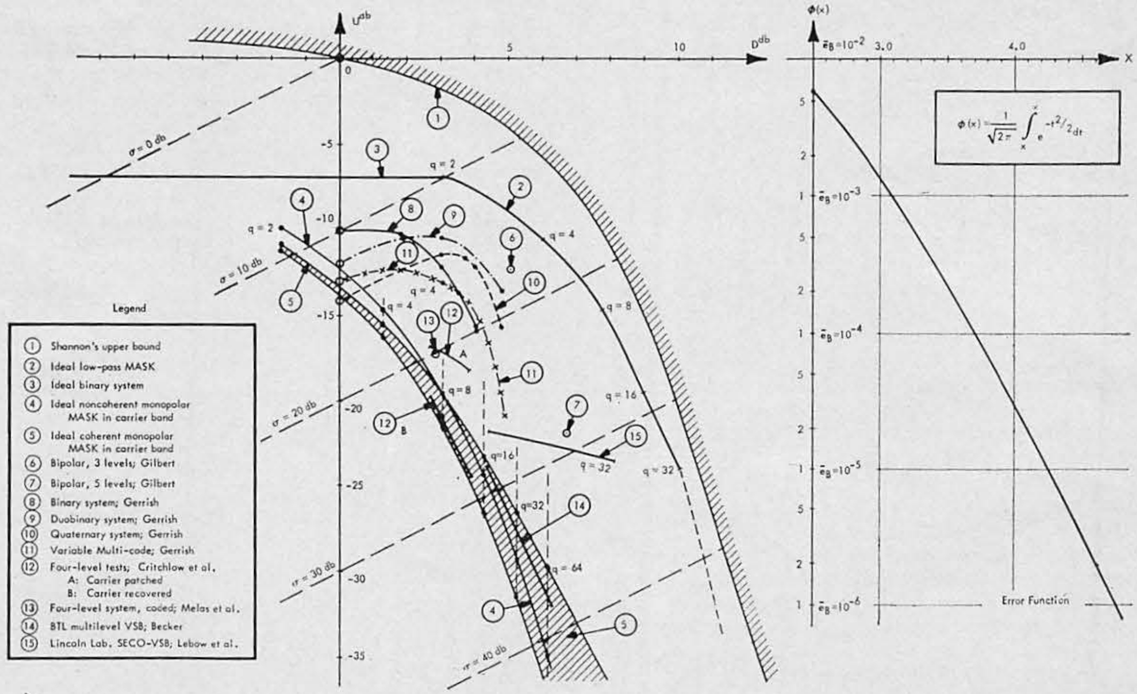


Figure 8.—(a) Utility curves of MASK systems.
(b) Error function.

according to our lower bound explained in Part I (equations 2 and 3). The number of levels (q in our case) has the same meaning as m in the Arthurs and Dym paper. The integral is tabulated in most mathematical table books. For the benefit of our readers, we plotted it as Q(x) in fig. 8b. Notice that this is related to the frequently used integral P(x) by the relation :

$$Q\{x\} = 1 - P\{x\}\,; \quad P\{x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz \qquad (27)$$

It can be seen that equation 26 is actually the equation for an error probability curve as discussed in Part I, section C3. To bring it in line with our notation, we also had to modify the energy contrast, which Arthurs and Dym referred to as

the energy per *character* E. We defined the energy contrast on the basis of the energy per output bit of information. As long as there is no redundancy in the transmission (synchronization characters etc.) and no coding involved, the relationship between character and bit energy contrast is simple :

$$\epsilon_c = \epsilon_B \cdot \log_2 q \tag{28}$$

Going from the error probability curve to the utility curve is simple when one remembers that the utility is the reciprocal value of the bit energy contrast (Part I, section C2). But one has not yet introduced the bit density, which requires a knowledge of the signal base. We recall from Part I, section C3, that the bit density is defined as the ratio of the transmission rate R_B in bits per second to the transmission bandwidth B_{TR}. Introducing the duration of one character T_c and assuming again that there is no redundancy, one can also express the bit density in the form :

$$D = \frac{R_B}{B_{TR}} = \frac{\log_2 q}{T_c \cdot B_{TR}} \tag{29}$$

Arthurs and Dym suggest in section G of their paper a value of 1.5 for the signal base (TB product) of each character. This gives us the final equation for relating the alphabet size q to the bit density :

$$D = \frac{\log_2 q}{1.5} = \frac{2}{3} \log_2 q\ , \tag{30}$$

$$D^{db} = 1.75 + 10 \log_{10} (\log_2 q)\ . \tag{30a}$$

It is now possible to plot the utility curve of Arthurs and Dym's ideal monopolar carrier-band MASK system. This is done in fig. 8a by curve 4. Remember that we plot utility curves in a logarithmic scale, i.e., both axes are calibrated in decibels. When plotting curve 4, one should recall that a utility curve is only valid for a fixed parameter of the error ratio. In the case of curve 4, we use $\bar{e}_B = 10^{-3}$ as the constant value which goes to the left side of equation 26. This equation must then be solved for u, using different values for q and taking the values of the function Q(x) from fig. 8b. Expressing the utility (u) in decibels (db) gives the ordinate value for each point of the curve. The corresponding D value (abscissa) results from equation 30a for each q.

Similarly, Arthurs and Dym's equation 107 may be converted to the magnitudes used in this paper. That equation gives an upper bound and a lower bound for the error probability of a monopolar MASK system with incoherent detection. The converted formula yields :

$$B(q) \log_{10} \{\bar{e}_B \log_2 q\} \leqslant u < B(q) \log_{10} \{\bar{e}_B \log_2 q + \log_{10} q\}\ , \tag{31}$$

$$B(q) = \frac{1.535\ .\ (q-1)\ .\ (2q-1)}{\log_2 q}\ . \tag{31a}$$

Notice that equation 31 gives the value for u. To be plotted in fig. 8a, it must be converted to decibels. The range between upper and lower bound is plotted as area 5 in fig. 8a. To get the corresponding values for D in db, one must use equation 30a, the same as in the previous case of coherent monopolar MASK.

A discussion of the questions raised by a comparison of these three ideal MASK systems may be useful to the reader. Curve 2 in fig. 8b represents an ideal low-pass system, having extremely low frequency components and using the tightest packing of waveforms one can imagine. Actually the composite of waveforms formed by a sequence of MASK signals according to this ideal system is an analog waveform ; and no practical system, so far, has been devised to code input information into such a composite transmission signal without distortions. No practical channels are known that would handle such waveforms without distortions. Thus one may expect that all practical MASK systems will have the utility curves below curve 2.

The ideal carrier-band MASK systems of Arthurs and Dym use waveforms that are easy to generate, i.e., rectangular transmission signals. This, at the same time, is why curves 4 and 5 are shifted so far to the left side of fig. 8a. Indeed, for using rectangular transmission signals, the authors pay a penalty of 1 : 3 (−5.67 db) in bit density against the low-pass system of curve 2, which operates at the Nyquist rate ($\beta_c = 0.5$). Even with this penalty, it is not reasonable to expect that no distortions would be inflicted upon the rectangular pulses. For example, a special message sequence would cause the transmitted waveform to alternate between maximum level and zero level, causing a sin x over x spectrum that extends far beyond the cutoff bandwidth of $1.5T_c$. The absence of these portions of signal energy which cannot be transmitted causes ringing in the receiver waveform that may easily exceed the threshold of the first level in a system with a high q value. The next subsection will review a number of research efforts aiming at more suitable waveforms for a one-dimensional MASK system than sin x over x or rectangular signals. Part IV of the paper will show how bandlimited orthogonal signal alphabets are still better suited to solve this problem for multidimensional MASK systems.

Another puzzling observation, when comparing the three ideal MASK systems, may be the small difference between coherent and noncoherent MASK systems. Curve 4 represents a coherent monopolar system—a highly redundant system, which does not make use of its redundancy. Once coherent reception can be secured, one may as well use all the negative amplitudes. Indeed, under assumption of a monopolar waveform alphabet, there is not much advantage in a coherent receiver over an envelope detector, although their analytical treatment is still considerably different, as shown by the different shapes of equations 26 and 31. It was the intention of the paper by Arthurs and Dym to point out this fact. Truly bipolar MASK systems will be discussed in subsection F6.

Yet another interesting observation is the fact that all these systems have their operating point for a given q value at about the same power contrast, (SNR) represented by the dashed sigma lines. Another very instructive comparison results when using equation 26 for the special case q = 2, representing a coherent binary AM system :

$$\bar{e}_B = Q\left\{\sqrt{\bar{\epsilon}_B}\right\} = Q\left\{\sqrt{\frac{1}{u}}\right\}. \tag{32}$$

This equation was originally derived by Reiger (53), Helstrom (55), and Lawton (58) who arrive at a general equation for coherent binary ideal systems introducing the correlation coefficient λ between the two waveforms representing Mark and Space respectively.

Considering further that equation 26 holds for unilateral MASK systems with zero amplitude level as the lowest level, we recognize that this zero amplitude level corresponds to $\lambda = 0$ in Lawton's ideal system. Thus equation 26 leads, in the limiting case of q = 2, to results identical with Lawton's ideal binary system. The fact that Lawton's system is derived for two waveforms of equal energy, while the MASK system of equation 26 uses one waveform of zero energy, does not matter because E in equation 26 is the average energy of all equally probable waveforms. Naturally, the highest utility for any binary system can be reached if the two waveforms have the maximum cross-correlation coefficient of $\lambda = -1$, which they reach if they are the negatives of each other. This leads to the equation for the best possible binary system, a phase-coherent system with identical waveforms of opposite polarity using a matched filter extractor.

$$\bar{e}_B = Q\left\{\sqrt{\frac{2}{u}}\right\}. \tag{34}$$

The utility of this optimum ideal binary system depends

only upon the acceptable error ratio. The utility can be read off directly from fig. 8b for any given error ratio. Assuming $\bar{e}_B = 10^{-3}$ we find $x = 3.10$; $u^{-1} = 4.81$ or $u = -6.8$ db. This value is plotted as the utility curve 3 of the ideal binary system in fig. 8a and also in fig. 1 of Part I as curve 3. It joins the ideal low-pass MASK system at the Nyquist rate ($D = 3$ db). Curve 3 is a straight line, demonstrating that the utility of the ideal binary system cannot be raised by reducing the bit density. Only nonbinary systems have the capability to trade bit density for utility. The binary system, however, can trade off power contrast versus transmission rate as mentioned in section C2 in Part I when explaining the utility chart.

F2. *Waveform Design in MASK Systems for Low-Pass Channels*

The comparison of the three ideal MASK systems clearly showed the need of a more sophisticated waveform design. Indeed, many researchers turned their attention to this problem. We shall first discuss the results for low-pass channels.

Gilbert (52) had already investigated many nonbinary alphabets. One of those alphabets has three levels (PA 1 in Gilbert's fig. 2) ; another one has five levels (PA 2). Gilbert refers to these waveform alphabets as quantized pulse amplitude modulation. The two operating points are placed into fig. 8a as points 6 and 7 respectively.

Hill (62) performed an intensive investigation of the influence of waveform distortions on the maximum number of amplitude levels that can be discriminated without error in a noise-free low-pass channel. He based his work on the theoretical fundamentals of pulse transmission published by Sunde (54). Hill defines a maximum error-free signalling rate (m.e.s.) and shows that this magnitude depends in a complicated way on the number of levels used for the waveform alphabet of a MASK system. Examples are given in which m.e.s. increases without limit ; passes through a peak ; remains constant or tends asymptotically to a constant value ; or decreases steadily, as the number of transmitted levels increases. Hill concludes that in a noise-free situation there is no large difference, whether one tries to increase the bit density by packing binary digits more densely than the Nyquist rate and permitting them to overlap, or to keep the digital rate below the Nyquist rate and pack more information into each digit by permitting more than two levels. In either case the designer learns to tolerate the resulting intersymbol distortions.

Lord and Lytle (64), recognizing that the ideal low-pass filter is unrealizable, nonetheless use it as a guide to novel techniques to solve the signal packing problems for low-pass channels. One of these is the pseudo-sample insertion (PSI) technique, which is actually designed to reduce the so-called aliasing error in sampled pulse transmission systems. In modified form it could be applied in MASK systems. A second technique suggested by Lord and Lytle is known as the vestigial interpolation filter (VIF) technique. Rather than attempting to eliminate the alias energy, this method makes use of it. It achieves the spectrum limitation in the frequency domain, while the PSI technique achieves the same goal in the time domain. The VIF technique is, therefore, closely related to the various vestigial sideband techniques of MASK transmission that we shall review in subsection F5.

Another group of research workers operates on the basis that the whole system, transmitter-channel-receiver, must be optimized to achieve the highest information density. A good guide for this approach is the paper by Tufts (65) which gives 38 references to earlier papers. Gerst and Diamond (61) demonstrated how conditions on the input waveform to a given system can be formulated to eliminate completely the output tail which normally causes the intersymbol interference. Petrovich and Razmakhnin (65) and Cory (60) investigated the distribution of the waveform energy in multilevel signalling systems.

F3. *Nonbinary Pulse Code Modulation Systems*

One important application of MASK methods is the pulse code modulation (PCM) systems. PCM was invented by A. H. Reeves (42) more than 25 years ago. It was intended to improve communications in two aspects : the first one is the noise immunity afforded by a digital system, as compared with an analog system (Goodall 47) ; the second aspect is the economy in equipment when combining PCM with synchronous time-division multiplexing as compared to frequency division multiplexing of many analog channels (Bennett 41). A third aspect, not suggested by the original inventor but frequently proposed since, is the possibility of achieving more efficient information packing through asynchronous multiplexing (Filipowsky 57). PCM is gaining increasing importance in modern short-haul (Davis 62) and long-distance (Litchman 63) communications trunks.

Ternary PCM is the most logical step to anticipate when extending binary PCM to multilevel PCM. In the passive ternary form there are three states : plus peak value, zero, and minus peak value. This passive ternary mode of operation has many interesting applications (Filipowsky and Scherer 56). It could be used to increase the bit density by a factor of $\log_2 3$ or for $+1.59$ db. There are many ways to organize the encoder of fig. 3 for translating the binary input into passive ternary waveforms of equal rate so that the transmission signal has no dc component and will meet a number of other requirements which differ from case to case (Aaron 62b). One possibility is *time polarity control* (TPC), first suggested by L. C. Thomas of the Bell Telephone Laboratories (BTL), where timeslots are labelled alternately positive and negative. If a unipolar pulse representing the binary *one* occurs in a positively labelled timeslot, it is transmitted unaltered. On the other hand, if a binary one occurs in a timeslot with a negative label, the pulse is transmitted with negative polarity. *Zeros* in the binary input are unaffected and will be transmitted as zeros in the passive ternary train. Such signals produce a power spectrum with discrete components at odd integer multiples of half the bit rate (Aaron 62a).

The *interleaved bipolar operation*, invented by M. Karnaugh, leads to still another restricted ternary mode (Aaron 62b). It is a combination of TPC with the bipolar mode and produces nulls at the integer multiples of a fraction of the bit rate. These interleaving modes of operation are particularly important in the short-haul applications of PCM, where it is necessary to use many bipolar repeaters in tandem (Mayo 62) and where the nonideal circuits and terminals in the exchange plant produce particularly severe limitations to the waveforms that can be handled (Shennum and Gray 62).

More recently, Sipress (65) of Bell Telephone Laboratories discussed a further improvement of the previously mentioned ternary modes, which he calls the *paired selected ternary (PST) transmission plan*. This plan improves the transmission of the timing information, permits monitoring of the lines during operation, and retains the elimination of the dc component in the same way as the other ternary coding procedures. It takes pairs of binary input digits and translates them into pairs of ternary transmission digits, thus offering a redundancy of 5 over 9 (more than 50 percent), which can be used in a variety of submodes.

Another extension of the bipolar codes to higher degrees is reported by Yazaki et al (65). These authors investigated a number of PCM translation procedures under the names of " pseudo-ternary code " and " bimode code ".

While the previously mentioned binary-to-ternary codes inserted redundancy to solve certain transmission problems, research also continues in the opposite direction (i.e., reducing the redundancy contained in the input information). From the several hundred publications in the field of redundancy

reduction (or information compression), we mention here only two that are directly related to nonbinary PCM: Farnqui and Das (64) reported tests of a "slope-quantized ternary" PCM system that operates in a range from 25- to 35-db power contrast with a pulse repetition rate of 20 to 60 KHz. Deregnancourt (65, 66) introduced the concept of variable length words for PCM transmission. It is a combination of time assignment speech interpolation (TASI) and conventional PCM. Basically it avoids the transmission of all nonsignificant zeros in the PCM words.

F4. *Special MASK Low-pass Systems*

In this subsection we shall discuss a further extension of the idea of translating the information from binary to nonbinary alphabets. The goal of the research reported below is the improved matching of the transmission signals to the channel characteristics, attacking the signal-shaping problem with the help of special code translation rules when going from binary to nonbinary codes. A first attempt in this direction is the binary to ternary code conversion (subsection F3). It suppresses the dc component. In the following paragraphs, the goal is the creation of a specially shaped power spectrum of the transmission signals, matching closely the frequency response of a telephone circuit or a general cosine-square-shaped frequency response.

The most publicized efforts in this direction are the research efforts of Lenkurt Electric Company, represented mainly by the papers of Lender. The duobinary technique (Lender 63a, b) is the first step to a more general polybinary concept (Lender 64a, b) or correlative level coding for binary data transmission (Lender 66). The basic idea is that of employing an encoder with memory, including the last K binary input digit in the encoding process for each new q-ary transmission digit. In the duobinary mode ($K = 2$) the last two binary input digits will be considered when forming the next ternary ($q = 3$) transmission digit. The digital rate of the binary input and the ternary transmission signal is exactly the same. This allows redundancy for the transmission waveforms. This redundancy, according to Lender's proposal, is used to shape the transmission waveforms in a way that they concentrate their energy into the lower frequencies, still avoiding, if required, any dc component. To achieve this goal he imposed on the encoder a translation rule that never permits a change in the q-ary levels of more than one step when going in time sequence from one q-ary digit to the next. This transmission rule makes sharp transitions impossible in the final transmission waveform of the MASK signal, thus avoiding strong energy in the high frequency end of the spectrum. Alternately, if this high frequency energy can be tolerated, the digital transmission rate can be raised, increasing the bit density of the system. Even with this restrictive translation rule, a certain amount of redundancy still remains in the q-ary transmission sequences, which can be used for a limited amount of error detection.

This correlative level coding method is specifically designed for non-ideal transmission channels. Its efficiency, accordingly, depends on the degree of matching between the resulting transmission waveforms and the actual channel characteristics. This is why no general computations of the utility of these systems are available and why it is difficult to compare them with other MASK systems.

Shagena and Kvarda (64) of Bendix Corporation designed an interesting encoder, performing an operation which they call "varilevel coding". It is also a binary to multilevel translator with equal input and output rate, which operates under the restriction that the q-ary output stays at the same level if the binary input is a zero, and changes for one step up if the binary input is a one. Naturally, after a certain number of ones, the translator output will have reached the highest level. The translator then reverses the stepping direction and reduces the level for one step whenever a binary one appears at the input. This continues until the lowest level is reached, when the stepping direction is again reversed. The result is a stepped zig-zag waveform, provided all q-ary output samples are held until the next sample is taken. This "staircase" waveform reaches the highest slope when the input consistently delivers ones. The system can claim that it will achieve a concentration of the waveform energy in the low frequency part of a low-pass channel, but it cannot avoid a dc component when the input consistently delivers zeros.

Gerrisch (65) of the Bell Telephone Laboratories compared the low-pass systems discussed above under simulated practical conditions. He used the attenuation and phase characteristics of a telephone line and assumed that a low-pass signal would be transmitted in coherent amplitude modulation over this channel. He added noise, simulated the various multilevel signals, and observed the eye pattern at the receiver after the signal had been contaminated by noise and distortions. He measured the loss in signal margin above the noise, as observed by the narrowing of the eyes, and plotted this signal voltage degradation as a function of the transmission rate. We converted his data to the magnitudes of the utility chart and plotted the results in fig. 8a. Curve 8, a two-level (binary) system, serves as a standard of reference. We started it arbitrarily at $u = -\sigma = -10$ db. The effective bandwidth of the low-pass channel described above was close to 1000 Hz. This point ($u = -10$ db; $D = 0$ db) corresponds to 1000 bits per second (bps). One can see that an increase of about 5 db in SNR is needed to operate this system at $D = 3$ db or 2000 bps (Nyquist rate for this channel), keeping it at the same error ratio (about 10^{-3}). Proceeding beyond the Nyquist rate causes such heavy distortions that the line drops rapidly. Curve 9 is the duobinary system that is clearly superior to all the other systems. It has its maximum utility at the Nyquist rate where it reaches the same utility value as the binary system at much lower rates. Naturally, far below the Nyquist rate, the binary system is superior. This is the power penalty one pays for the multilevel mode of operation. The utility curve indicates that the payoff for this penalty is a fair return in increased transmission rate.

The comparison with the other two multilevel systems shown in fig. 8a is not so favourable, although at highest rates all systems are better than the binary system. Curve 10 represents a four-level MASK system without any coding for wave-shaping. It clearly shows a maximum at the Nyquist rate. Beyond this rate, intersymbol interference takes its toll and the curve falls more rapidly than the duobinary system. The varilevel coding system (curve 11) has the lowest utility. It has its own maximum utility at about 70 per cent of the Nyquist rate, but it stays several decibels below the other systems.

Finally, we refer to a special PCM system published by Brogle (60) of Radio Corporation of America. He arrives at a ternary system by arithmetically adding two binary channels that are assumed to operate synchronously. If both channels have a Mark, his adder produces a positive output level; if both channels have a Space, the adder produces a negative output level of the same absolute value. If one channel has a Mark and the other a Space, the adder stays at zero level. To resolve the ambiguity of this last case, Brogle shifts the channels against each other in phase for one-half elementary length. The resulting biternary output wave is said to transmit a maximum rate of information with a minimum number of pulse amplitudes.

In summarizing, it may be stressed that a number of ingenuous ideas have been published on the use of MASK methods to match binary information to practical channels, and it can be anticipated that some of these methods will gain importance when used in combination with suitable modulation, coding, or multiplexing systems.

F5. *Special Voiceband Systems (SSB and VSB)*

Vestigial sideband modulation (VSB) is an amplitude modulation system that completely transmits one sideband

except for the lowest frequencies, which are linearly tapered off to 50 percent of their normal value at the carrier frequency, while the other sideband is completely suppressed except for the lowest frequency components, which are raised linearly up to 50 percent at the carrier frequency. This treatment secures transmission of the lowest frequencies in a quasi-double-sideband mode and it also ensures the transmission of a dc component. This is very important for the undistorted detection of pulsive waveforms (Murakami and Sonnenfeldt 58).

The Sebit 25 of Rixon Electronics Inc. is most likely the first commercial VSB data terminal (Holland and Myrick 59a, b). It was designed in the years before 1959, was transistorized, and equipped with plug-in modules to adapt it to various operational situations. It was designed for 2500-bits-per-second transmission rates and used a carrier at 2500 Hz. It provides manual adjustments to compensate independently for amplitude and phase (or time delay) distortions which the signals may suffer when running over typical telephone circuits. A fast-acting automatic gain control (AGC) system was employed to accommodate variations of more than 50 db in the level of the receiver input signals. A rather complex timing system was needed to ensure accurate resampling of the mutilated waveforms extracted by the VSB receiver.

Extensive further development work in the last eight years added many improvements to the terminal publicized in 1959. Groff and Powers (61) of ACF Industries Inc. described the design and operation of a " synchronous 4800 bps data terminal that utilizes quaternary, AM, suppressed-carrier vestigial sideband transmission ". This equipment employs a two-step encoder : in the first step, the binary input signals are changed into a form in which the information is carried in the transitions from one level to the other, rather than in the levels themselves ; in the second step, the signals resulting from the first step are converted into quaternary samples. This binary-to-quarternary conversion may be bypassed when operating at 2400 bits per second. The quaternary samples are preshaped by a shaping filter and then modulated in a balanced modulator to a 2400-Hz carrier which is synchronized to the input data rate (via 2 : 1 or 1 : 1 divider respectively). The output of the balanced modulator is run through a vestigial sideband filter. The receiver uses an interesting arrangement with a special inter-symbol-interference corrector (ISIC) consisting of 16 stages of delay elements, the outputs of which are summed in a special adjustable network (Gibson 61a, b).

The development of a vestigial sideband phase-reversal data transmission system is reported by International Business Machines Corporation (McAuliffe 62, Critchlow 64). This system employs an improved method of carrier recovery for VSB systems which frees the designer from the constraint that the bit rate and carrier frequency have to be synchronized. A terminal of this system can, for example, accept teletypewriter signals in place of synchronously clocked binary data. The carrier recovery system uses a complementary VSB filter that has, in the vicinity of the carrier frequency, the opposite frequency response of the original VSB filter used in the transmitter. Together the two filters form a narrow bandpass characteristic of half-cosine shape with maximum response at the carrier frequency. The rest of the carrier recovery circuitry is identical to the usual double sideband (DSB) carrier recovery systems with full-wave rectifier and an optional phase-locked oscillator. The IBM team performed functional tests with a four-level mode of this VSB data system. These tests were conducted in the laboratory over the carrier channel, the characteristics of which are not specified in detail. Assuming a bandwidth of 3300 Hz for this carrier channel, one can plot the results in the utility chart (fig. 8a). Curve 12A corresponds to the utility of the system if the data carrier is patched through between transmitter and receiver. Curve 12B applies when the carrier is recovered in the receiver from the noisy signal. Noise from a random noise generator was added at the input to the carrier channel. Both curves are for an error ratio of 10^{-5}, the only value given by Critchlow. Field tests were performed with 4000 bps over a long distance line and with 8000 bps over a signal carrier link. Eye diagrams are supplied as the results of these tests.

The VSB system described above was subsequently used to test the technical feasibility of automatic distortion compensation (Schreiner et al 65). The results of these tests were encouraging, but the authors kept the possibility open that " systematic approaches to automatic time domain equalization other than those discussed [by them] may lead to even better results ".

Another team of IBM researchers from France described an interesting improvement which amounts to a combination of the wave-shaping binary to q-ary coding procedures discussed in subsection F4 and a SSB modulation method (Melas and Gorog 64). This system has an encoder that uses four binary elements (α_1, α_2, α_3, α_4) and converts them into four quaternary elements (β_1, β_2, β_3, β_4), each selected from an alphabet (A, B, C, D) of four bipolar amplitude levels (for example : $+3$, $+1$, -1, -3). The encoding rules are the following :

$$\begin{aligned} \beta_1 &= \alpha_1 (\alpha_3 + 2) \\ \beta_2 &= \alpha_2 (\alpha_4 + 2) \\ \beta_3 &= -\alpha_1 (\alpha_3 + 2) = -\beta_1 \\ \beta_4 &= -\alpha_2 (\alpha_4 + 2) = -\beta_2 \end{aligned} \tag{35}$$

Naturally, as the rate of the binary input is the same as the rate of the quaternary output, the encoder introduces redundancy. This redundancy is used to eliminate the dc component ($\sum_1^4 \beta_i = 0$) and to shape the spectrum so that the highest essential frequency component is one half of the digital rate. These quaternary output digits of the encoder are held for one elementary duration each so that they form a sequence of rectangular steps. This quaternary step waveform is then passed through a low-pass filter with about 2500-Hz cutoff before it is submitted to the SSB modulator, where it is modulated to a carrier of 2800 Hz. The lower sideband is separated by a low-pass filter of 3000-Hz cutoff frequency and the output of this filter is used as the transmission signal. The result of this combined encoding and modulation process is a nonbinary transmission signal with a spectrum well matched to a telephone channel and with the double advantage of having no dc component and no information dependent component at the carrier frequency of 2800 Hz. A rest carrier is purposely transmitted and can easily be used in the receiver for synchronous demodulation. There is no need to lock the data rate to the carrier frequency, but there is also no ill effect if this is done for the sake of simpler synchronization. The receiver decoder uses a resampler which samples only the β_1 and β_2 positions, but neglects to sample the β_3 and β_4 positions. One of the latter ones shows intersymbol interference after the shaping, modulation, and demodulation processes. Eye diagrams demonstrate the small amount of intersymbol interference at the correct sampling instance when operating the system with 4800 bps. The authors claim that, at this rate, tests over 4B-type telephone lines yielded 10^{-5} error ratio for 20-db power contrast (SNR). This places the system at operating point 13 (D = 2.67 db ; u = -17.3 db) in fig. 8a.

McAuliffe (64) of North American Aviation Inc. describes a simplification of the VSB transmission system discussed by Critchlow et al (64). This simplification concerns the VSB filter at the transmitter, which now can be designed without delay equalization. This has become possible

because of the application of a novel method of automatic data equalization, which equalizes the data transitions dynamically. The static amplitude and phase characteristics are ignored ; instead, attention is focused on the dynamic time response of the channel to data transitions. All modem (modulator-demodulator) filters may be considered as part of the channel and the complete complex is equalized at once. In this technique the equalization is actually matched to the specific signal used, which is why it is termed " data equalization ". The goal of the technique, which operates on a digital basis, is shaping the input-digits, together with interlaced dummy digits, in such form that the output at the receiver gives the largest signal change at the decision moment for any input data change. The technique is described by the author with reference to digital pre-distortion in the transmitter, but he later extends his discussion to post-detection equalization. The terminal incorporating this improvement is called Adaptively Data Equalized high speed Modem (ADEM). It is claimed that up to 5400 bps can be transmitted when combining ADEM with a VSB modulation mode.

The most refined development of VSB data transmission equipment is in progress in the Bell Telephone Laboratories. Starting with a 1650-bps system (Brand and Carter 62), released around 1961, staff members of this organization described improved VSB equipment with the capability to transmit asynchronously data rates at 2000 bps over the switched telephone network and at 3000 bps over equalized voiceband circuits (Becker et al 62). A later release from the same laboratories describes an experimental multilevel VSB terminal for 2400, 4800, 7200 and 9600 bits per second (Becker 65) operating with 2, 4, 8 and 16 levels, respectively. Becker concludes that the following conditions must be met when designing an optimum data transmission system :

(1) Choose the modulation system with the best bandwidth utilization.

(2) Equalize delay and the amplitude characteristic of the channel automatically so as to exploit its full bandwidth.

(3) Couple the system with error correction so as to meet the accuracy requirement of the system while also maximizing the data rate.

It is interesting to notice that these three conditions are in line with the cardinal principles No. 4, No. 2, and No. 6, part of the ten cardinal principles to be followed for the optimization of an integrated data transmission system (Filipowsky 59).

To meet the three conditions postulated above, Becker (65) explains that the best design involves a multilevel VSB modulation mode, described in his paper ; an automatic equalization system described in a paper by Lucky (65) ; and an error control system described in a paper by Burton and Weldon (65b).

The combined effect of the application of these three principles is an integrated data system that can operate up to 4.1 db bit density at -23.9 db utility and up to 6.3 db bit density at -32.2 db utility. This operating line is placed as line 14 into fig. 8a. Using fig. 8 from Becker (65) and using the average line characteristics for background noise from Alexander et al (60), the systems performance has been calculated in the units for our utility chart, assuming an average minimum signal strength at the receiver input of -32 dbm. This gives a power contrast of 28 db for long-distance lines and 38.5 db for short-haul lines. The effective bandwidth is assumed at 2000 Hz from fig. 2 of Becker (65). Pulsive noise is not directly considered in the determination of the power contrast but indirectly by plotting line 14 for an error ratio of 10^{-5}, keeping in mind that the system operates with a complex error-correction system. It is interesting to note that curve 14 falls right into the theoretical range predicted by Arthurs and Dym (62) for a coherent MASK system. Notice, however, that the theoretical model is for monopolar multilevel signals in DSB ; it is plotted for an error ratio of 10^{-3} in white noise ; and it does not assume error correction. The practical system of the Bell Telephone Laboratories team, as plotted for 10^{-5} error ratio, operates over a variety of circuits under natural noise conditions, employs VSB, and uses an error-correction system. Despite these differences it is significant that the results of the theoretical analysis and of the operational tests of a MASK system fall into the same small area of our utility chart. It is surely an indication of the high engineering standard of this terminal, which operates at twice the Nyquist rate.

This remarkable achievement of the Bell Telephone Laboratories terminal is due to a new signal format with two pilot carriers at 600 Hz and 3000 Hz (Becker et al 66), a very sophisticated automatic equalization system (Lucky 65a, b, Becker et al 65), and an efficient error control system (Burton and Weldon 65a, b). The carrier and timing recovery subsystem is a design with a previously unbelievable precision of operation. Figure 9a shows the eye pattern of this terminal in the binary mode (2400 bps) and in the octonary mode (7200 bps), along with a raised cosine pulse waveform. Figure 9b demonstrates the improvement due to equalization with the automatic equalizer described by Lucky (65a, b).

F6. *Carrier-Band Systems*

While VSB is an excellent modulation method for matching digital signals to the special characteristics of a voice-band channel, other methods may be preferable for general types of carrier-band channels like those used in radio links.

The advantages of double sideband transmission, primarily in the suppressed carrier mode, have been stressed in many publications (Costas 59, Gibby 60, Tsikin 62). The previously mentioned varilevel coding method of Bendix Corporation (Shagena and Kvarda 64) is used in connection with a DSB modulation method, placing the carrier at 1800 Hz into the centre of a voice-band. A balanced modulator and a synchronous demodulator are used to handle the suppressed carrier signals.

The Philips Research Laboratories worked on the development of "co-modulation", a new method of high-speed data transmission (DeJager and Van Gerwen 62). The name stands for complementary orthogonal modulation. The primary object is to operate at higher bit densities than standard systems. The essential features of the system involve :

(1) Suppression of the lower frequencies in the base-band signals before modulation with the help of a high-pass filter.

(2) Multiplexing of two binary channels, by using two carrier voltages differing 90 degrees in phase.

(3) Restoring in the receiver the frequency components, originally suppressed in the transmitter, with the help of a feedback network.

Feature 1 enables the system to insert a pilot signal at the carrier frequency for phase reference and for coherent detection. Feature 2 causes the transmission signal to look like a quaternary phase-modulated signal when both channels are keyed synchronously.

An experimental system operated with a rate of 4000 bps over a normal carrier-telephone channel in a band from 700 Hz to 3100 Hz, yielding a bit density of 1.67. Phase equalization was necessary over certain kinds of lines. O'Neill and Saltzberg (66) described an automatic equalizer for this class of coherent quadrature-carrier data transmission systems.

F7. *Other MASK Developments*

In addition to the special development efforts reported in the above subsections, Bason (66), in Sydney, designed a transistorized experimental model of a multilevel system

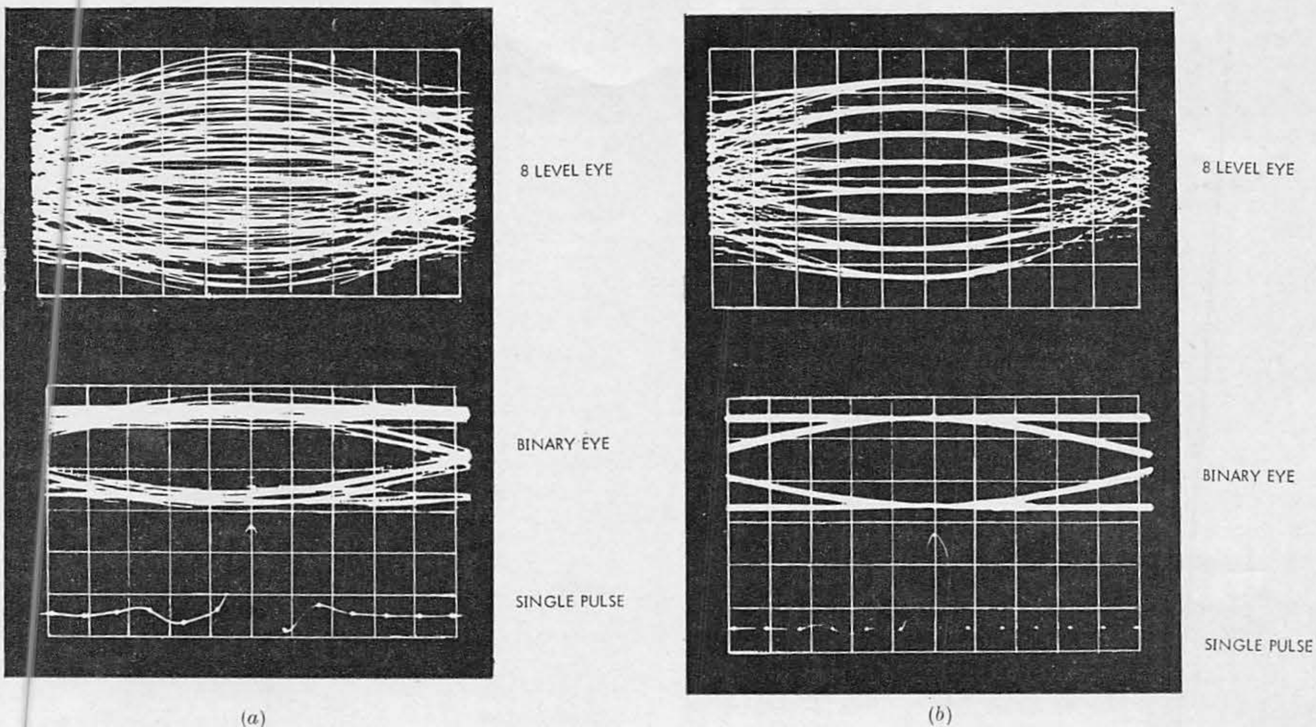


Figure 9.—(*a*) *Unequalized operation.*
(*b*) *Equalized operation.*
(*Courtesy of R. W. Lucky, Bell Tel. Lab. and IEEE, New York.*)

for the simulation of various signal, channel, and noise combinations. This experimental setup permits the setting of linear or nonlinear threshold adjustments and provides for the simulation of many message types with rates up to 4000 bps.

Litchman (64) of ITT Communications Systems Inc. examined the technical prospects for handling large volumes of digital traffic over submarine cables. He develops arguments for an all-digital cable and describes in detail an economical approach using existing repeater and cable technology for the transmission of multilevel digital data at rates in excess of 1.5 megabits per second.

Lebow et al (63) of Lincoln Laboratory, Massachusetts Institute of Technology, report another ultra-high rate experimental data system of similar integrated character as the Bell Telephone Laboratories system described previously. It is a truly adaptive variable rate system and it combines the following features :

(1) The encoder is a self-regulating error-correcting coder-encoder (SECO) (Wozencraft and Reiffen 61, Perry and Wozencraft 62), which can operate with redundancies of eight different values, of which three were used in the tests reported here : $3/5$, $2/5$, $1/5$. It is a system that takes a long sequence of input bits (60 in this case) and inserts an average of $(1/r - 1)$ parity check bits for each data bit, with r assuming the above values of redundancy. From the received data bits, the decoder generates the same sequences that the coder would generate had it actually received the decoded message as input. For each newly received bit, the decoder attempts to find that hypothetical input message that agrees sufficiently well with the one resulting from the decoded noisy message. This process is updated on a sequential basis so that a cluster of errors may be corrected if it is followed by a sufficiently long string of correct bits before the 60 bits are reached that constitute the duration involved in the process.

(2) The encoder also contains a translator to a $q = 32$ multilevel alphabet. Assume that the input of the Lincoln Laboratory system operates at the highest rate of 9000 bps and that $r = 3/5$ is selected. This means that, in the translator, the encoder handles $9000 + 5400 = 14{,}400$ bps. The output of the translator will submit $14{,}400/\log_2 32 = 2880$ multilevel signals to the modulator.

(3) First, the modulator has a low-pass pulse-shaping network, which generates with the help of 5 microsecond pulses and a tapped delay line and output low-pass waveform with precise nulls at intervals corresponding to the digital rate (2880 Hz in this case). This signal-synthesis network produces a low-pass spectrum which best matches the telephone line characteristics when translated into a VSB signal with the help of the actual modulator, using a carrier at 2500 Hz.

(4) A pilot tone at 3000 Hz, together with the carrier, ensures phase-correct synchronous demodulation even in the presence of a phase crawl over the carrier telephone system.

(5) The receiver demodulator has a very sensitive carrier-extraction system and operates as a synchronous demodulator.

(6) The decoder operates as described under 1 and measures its own ability to decode rapidly and efficiently. This can be done by noticing the decoding distances and the buffer delay during operation. When these values increase too fast the receiver will request over a return channel a slowing down of the rate and/or an increase of redundancy. When the opposite is the case, the receiver will request an acceleration of the operation.

This SECO-VSB system operates practically error-free up to its breakdown point. The authors report that an experimental system was operated in 1963 over an 800-mile

loop of a type K carrier system with throughput rates from 6000 to 9000 bps. Using average values of the power contrast for such circuits places this MASK system at a higher utility and bit density than the Bell Telephone Laboratories system of similar structure. Notice, however, that the SECO-VSB system needs a return channel and that the tests reported by Lebow et al (63) were conducted with a noise-free return channel.

Concluding the section on MASK systems it may be stated that a large variety of developments are in progress and that this class of systems seems to be well qualified to fill the upper part of the right side of the utility chart. The MASK-VSB combination seems to emerge as the most efficient solution for this class of nonbinary systems.

G. Multifrequency Shift Keying (MFSK) Systems

G1. *Fundamentals of FSK and MFSK Systems*

The same reasons that made frequency modulation (FM) so popular are also the reasons for the wide application of its digital counterpart, usually known under the name frequency shift keying (FSK). It has long been known as a carrier telegraph method ; radio telegraphy used this mode of operation for several decades, benefiting from the low sensitivity to fading and noise which characterizes particularly the wide-band FSK mode (Jones and Pfleger 46, Davey and Matte 48, H. O. Peterson et al 46).

Conventional radio telegraphy operates with two states : Mark and Space. In the FSK mode each state corresponds to a special constant frequency. The receiver has the *a priori* knowledge of these two frequencies and its task is to extract the signals from noise and to decide which of the two frequencies has most likely been transmitted in a given interval. Naturally, to avoid confusion, these two frequencies should be so far apart that filters tuned to one would not be affected by the signals shifted to the other frequency. On the other hand, one would like to keep them close enough together that the combined spectrum of Mark and Space signals would have the minimum band occupancy consistent with the desired digital transmission rate. In terms of the definitions in Part I, this means that the signal base β would have the minimum value for operation with acceptable signal distortions, for binary FSK β is usually between 1.2 and 1.8 (Feldman and Faraone (61). Multifrequency shift keying systems are basically an extension of the binary FSK systems. More than two frequency states are used at the transmitter and the receiver has the more difficult task of making a nonbinary decision. Figure 4 showed the principal elements of an MFSK system using eight frequencies to transmit three bits of information with each pulsive signal. Many combinations of transmitter methods and receiver methods are conceivable for FSK systems.

Figure 10 shows on the left side three different arrangements of the transmitting side of an FSK terminal. On the top, bottom, and right side are six arrangements for the receiving side. Principally, each transmitting block diagram could be combined with any one of the six receiving block diagrams to form a workable system. Yet certain combinations are more efficient than others. Those preferred in practice are indicated by shaded areas in the centre of fig. 10.

The combination T_1-R_1 is possibly the first FSK system that has been brought into operation. It simply performs exactly what has been defined above as FSK. There are two oscillators operating on F_1 and F_2, but two mutually exclusive analog gates (or one relay) permit only one of the two frequencies to reach the transmission channel. The binary source controls these gates. The receiver R_1 has two bandpass filters centred on F_1 and F_2 to separate Mark from Space signals. If only binary signals are to be received, the receiver will preferably operate in a clocked mode with a synchronizing circuit triggering the decider at the end of each digital interval (Glenn 66). This combination T_1-R_1 is sometimes known as *noncoherent FSK*, but we prefer the term *switched FSK*, reserving the term noncoherent for any FSK method applying the receiver arrangement R_1 or R_4 but not necessarily using the transmitter arrangement T_1. A system may be coherent on both ends, on either end, or nowhere. Little attention is given in the literature to this ambiguity of the term *coherent*, though Bennett and Rice (63) discriminate clearly between waveforms with discontinuous phase (case T_1) and waveforms with continuous phase (case T_3), while many authors discriminate between coherent FSK and noncoherent FSK, assuming continuous phase on the transmitting side but coherent or noncoherent detection on the receiving side. The output of the arrangement T_1 can be made continuous when the difference between the two frequencies is made an exact integer multiple of the binary input rate, assuming perfectly constant frequencies and instantaneous switching action.

The combination T_1-R_2 can lead to a coherent FSK system if the local oscillators in the receiver can be synchronized

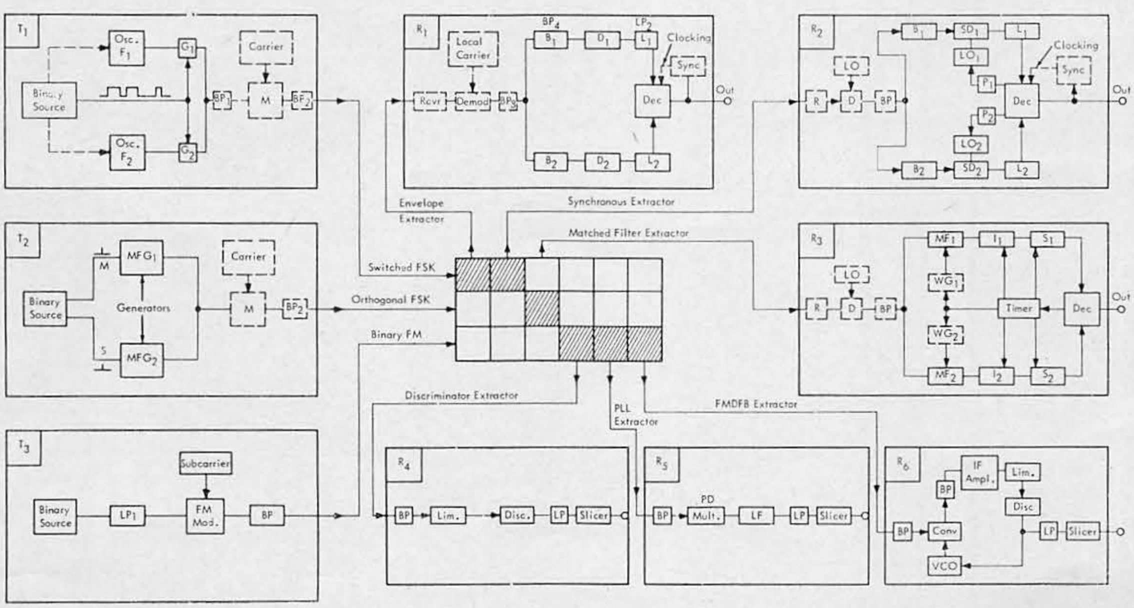


Figure 10.—Circuit arrangements of binary FSK systems.

and correctly phase locked to the respective oscillators in the transmitter.

Any of the combinations mentioned thus far can be used in connection with the optional carrier modulators. Such systems frequently carry the designation FSK-XM, where X stands for the kind of carrier modulation applied. The most frequently used system is FSK-AM, primarily FSK-SSB (Lender 65b, Glenn 59, 60). FSK-FM is usually discussed as a pulse code modulation system and designated PCM-FM-FM, where the first PCM-FM is closely related to the third preferred FSK combination to be discussed below in connection with arrangement T_3.

An important combination is T_2-R_3. This is generally called the orthogonal FSK system, although the arrangement T_1 can also produce a pair of orthogonal waveforms when enforcing certain ratios between the bit rate, F_1, and F_2. Some authors call any system orthogonal which is free of cross talk between Mark and Space waveforms, thus tying the orthogonality to the receiver extractor characteristics. The advantage of the combination T_2-R_3 is that it is theoretically optimum, and that it offers the designer the possibility of shaping the amplitude of the FSK waveforms for matching them to a bandlimited channel.

The remaining preferred combinations are systems using T3 as transmitter terminal and either R_4, R_5, or R_6 as receiver terminal. The block diagram for T_3 basically shows an analog FM modulator that receives a low-pass waveform as information carrier. The resulting signal is usually designated as binary FM. This arrangement ensures a waveform without phase discontinuities. A dc component can be removed by introducing into the base-band signal a rectangular carrier of the bit rate, frequently called the clock.

The three preferred receiver arrangements R_4, R_5, R_6 show significant differences. However, they all have in common the improvement threshold. In the terminology of Part I of this paper we may say that the binary FM systems call for a nonlinear signal processor (Part I, section C3). R_4 is the straightforward FM receiver arrangement and, indeed, it is frequently applied in binary FM (Lender 65b, Glenn 60, Bennett and Salz 63b, Zabronsky 61, Meyerhoff and Mazer 61, Klapper 66). All these publications discuss the various aspects of the threshold problem. Schilling et al (65) pointed out that the nonlinear extractors cause rather damaging noise spikes in the output of a limiter-discriminator circuit, even if the input is contaminated by Gaussian noise. This effect seems to be the reason that the error rates of actual measurements differed considerably from the calculated values.

The receiver arrangement R_5 uses a phase lock loop. Schilling et al (65) compares this arrangement with R_4. It is shown, by using Schilling's equivalent circuit (Schilling and Billig 64), that, by proper adjustment of the transmission parameters (frequency deviation and data rate), errors due to spikes can be made equal to zero. This result was experimentally verified with a receiver arrangement of the type R_5.

The receiver arrangement R_6 finally employs the FM demodulator with feedback (FMDFB) loop, which recently has been much used in space communications. Its analysis is similar in complexity to the analysis of the phase lock loop, but Schilling and Billig (64) extended Rice's technique (Rice 63) to these circuits. They achieved results which are valid above, at, and below threshold and which are easily obtained because of the simplicity of the equivalent circuit. Other references to FMDFB circuits may be found in Van Trees (63), Ruthroff (62), Enloe (62), and in Giger and Chaffee (63).

The MFSK systems may be subdivided roughly into the same three classes as the binary FSK systems, although many hybrid systems are also conceivable in the MFSK case. This classification leads to an interesting map of " spheres of influence " of the various classes when marking their general operating areas in the utility chart (fig. 11). On the far left wing (lowest bit density) are the large TB (wide-band) FSK systems (section G4). The range from -4 to -20 db bit density is the domain of the small TB systems of subsection G3. The centre belongs to the binary FSK systems. The extreme right (i.e., the area of high bit densities) can be reached only by MFSK systems of the class designated as MASK-FM (subsection G5). They are hybrid systems and operate with extremely small FM deviation ratio. Thus, we see that theory does not concede them an advantage over direct MASK systems. Indeed, the typical utility curve for MASK-FM systems will lie below the corresponding MASK curve, notwithstanding some practical advantages that the MASK-FM system may offer in special cases.

Concluding this subsection on MFSK fundamentals, we would like to stress the fact that fig. 11 shows the versatile character of MFSK systems ranging over five to seven decades of bit density and offering reasonable, though not optimum, utility. Notice, however, that fig. 11 is merely a schematic diagram in which only the large TB utility curves are taken from actually published values (Glenn 66), while all the other utility curves are typical representatives of their class without reference to any special investigation. Accurate

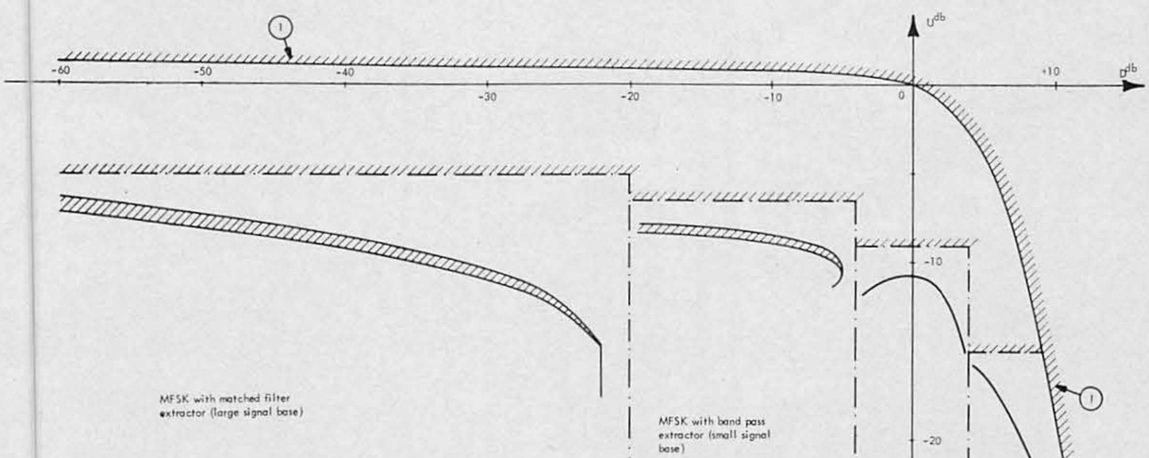


Figure 11.—Typical utility curves of various MFSK classes.

utility curves for the other classes will be given in fig. 12 and will be explained in the following subsections.

G2. *Ideal MFSK Systems*

It is again convenient to start with the ideal binary FSK systems. We showed in fig. 1, line 3, the utility curve of the ideal binary system, which has been repeated in fig. 8a. Section F1, equation 34, shows that the ideal binary system has a constant utility for each value of binary error ratio. For $\bar{e}_B = 10^{-3}$, this value is -6.8 db. The ideal binary system requires a correlation coefficient of -1, which can only be reached with two waveforms being the negative of each other. FSK waveforms must have a cross-correlation coefficient larger than -1, with 0 being a preferred value in many cases (orthogonal binary FSK). Under ideal conditions they should approach the value of formula (33) for $\lambda = 0$; i.e., their utility curve should be governed by the equation :

$$\bar{e}_B = Q\left\{\frac{1}{\sqrt{u}}\right\} . \tag{36}$$

This is again a straight line when plotted for constant $\bar{e}_B$.

contrast. Under these assumptions, Lawton arrives at the following equation :

$$\bar{e}_B = \frac{1}{2} \cdot e^{-1/(2u)} \tag{37}$$

Solving for u gives :

$$u = -\frac{1}{2} \cdot \frac{1}{\log_e 2\bar{e}_B} \tag{38}$$

or in decibels :

$$u^{db} = -3.01 - 10 \log_{10} [-0.693 - \log_e \bar{e}_B] . \tag{38a}$$

For the special case $\bar{e}_B = 10^{-3}$, one can find $\log_e 10^{-3} = -6.908$ and $u = -10.9$ db. Again it can be seen that the binary utility curve is a straight line. It is plotted as line 12B in fig. 12.

Turning from binary mathematical models to MFSK models, we see in fig. 12 the utility curves of two different models, published at about the same time. From Arthurs and Dym (62), we derived curves 6A and 6B ; Akima (63) published, apparently independently, equations and charts for a slightly different MFSK model. Akima's results are plotted as curves 5A and 5B in fig. 12.

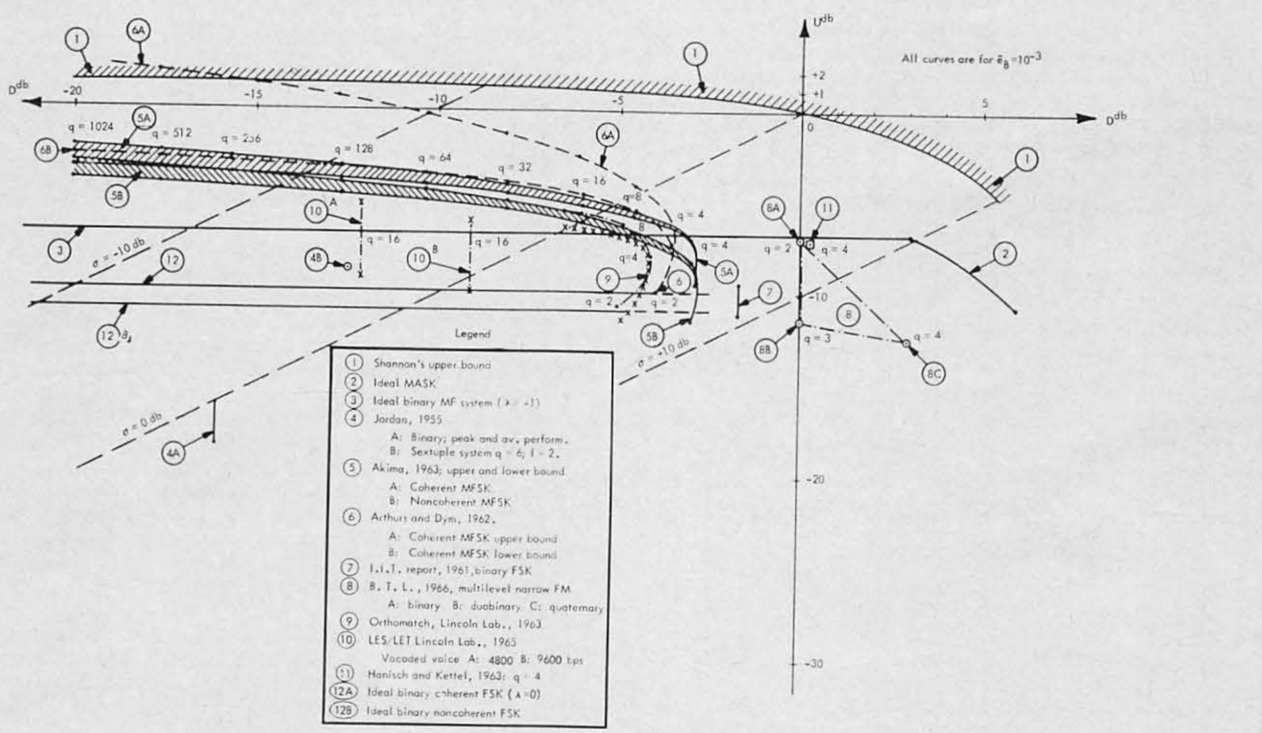


Figure 12.—Utility curves of MFSK systems.

Using our favourite value $\bar{e}_B = 10^{-3}$ leads (with the help of fig. 8a) to $u = -9.9$ db. This result is 3 db less than the ideal binary system, a fact which may be verified when comparing equations 36 and 34. To act as a wide-band system and to maintain $\lambda = 0$ requires that D should not be larger than about -4 db. This ideal binary coherent FSK system is shown in fig. 12 as curve 12A along with curves 1 (Shannon's upper bound) and 3 (ideal coherent binary matched filter system, equation 34), and with many others that will be explained throughout this section.

Lawton (58) derived the equation for the error ratio of an ideal non-coherent FSK system with matched filters after envelope detectors. This corresponds to the receiver arrangement R_1 in fig. 10, with the low-pass filters L_1 and L_2 acting as matched filters. This ideal system also assumes that the decider will sample the filter outputs precisely at the end of the digital interval and that the envelope detectors offer ideal operation, also at a very small power

Arthurs and Dym give an equation (number 64 in their paper) for the error ratio of a coherent MFSK system operating with matched filters or correlators. They derive the character error ratio that we had to convert to the bit error ratio to get the utility curve. We discussed this problem in section C1. For the present purpose we took the lower limit of equation 2 and converted e_c to e_B, arriving at :

$$\frac{1}{\log_2 q} Q\left\{\sqrt{\frac{\log_2 q}{u}}\right\} \leqslant \bar{e}_B \leqslant \frac{q-1}{\log_2 q} \cdot Q\left\{\sqrt{\frac{\log_2 q}{u}}\right\} . \tag{39}$$

This equation for a fixed parameter $\bar{e}_B$ $(=10^{-3})$ and for each value of q gives a value of u. To plot a utility curve one also needs the value of D as a function of the number of frequencies (q). Arthurs and Dym recommend a bandwidth law for this purpose (their equation 133) which, in the magnitudes of this paper, can be expressed as :

$$D = 2 \cdot \frac{\log_2 q}{2q + 1} = \frac{\log_2 q}{q} \cdot \frac{1}{1 + 1/2q} . \tag{40}$$

With the help of equations 39 and 40 it is now possible to plot curves 6A and 6B as upper and lower bounds of the utility of an ideal coherent MFSK system. The special case $q = 2$ leads to $u = -9.9$ db and $D = -4.0$ db, a point which lies also on curve 12A, the utility curve for the ideal binary coherent FSK system. This point is the starting point for both the bounds of equation 39. The utility curves 6A and 6B reach the highest bit density for a quaternary MFSK system and the utility monotonously increases with the increasing size of the MFSK waveform alphabet. The bit density, however, rapidly declines beyond octonary systems. Comparing nonbinary FSK with binary FSK shows that the ideal nonbinary system offers at least 7-db improvement in utility over the binary system when using 128 frequencies and reading the value from the lower bound of the utility curve. The upper bound of the utility curve is represented by curve 6A and results from the lower bound of equation 39. This utility curve apparently is overly optimistic, since we see that it crosses the upper bound of the channel capacity at $D = -15$ db. The lower bound, curve 6B, was derived earlier by Helstrom (60) and it falls well within the range of Akima's model.

Curves 5A and 5B relate to another model of a coherent MFSK system which differs only slightly from the Arthurs and Dym model. It is based on the " theory of the m-positions experiment ", resulting in an integral for the probability of success of the experiment (Lawson and Uhlenbeck 50).

Using tables or computers, one arrives at error curves of the form shown by fig. 1 in Akima's paper (Akima 63). Akima's charts, like most of the calculations and charts in the literature, are for the q-ary digital signal error probability and for the digital energy. We converted Akima's charts to bit-error probability ($\bar{e}_B$), using for the first purpose both limits of equation 2. The result is plotted in fig. 12 as area 5A. The curves indicate that the operation of the decoding system has no great influence on the utility. Whether the decoder makes, on the average for each m-ary error, only one binary error or m/2 binary errors (equation 2) changes the utility curve only from the upper limit of area 5A to the lower limit of area 5A.

When plotting the curves 5A, we used Akima's assumption about the band occupancy in MFSK systems. His assumption leads to the following equation for the bit density :

$$D = \frac{\log_2 q}{q} . \tag{41}$$

It can be seen that his is a special case of equation 76 of Arthurs and Dym for the value $q \rightarrow \infty$. The two equations differ most for $q = 2$; this is the reason that Akima curves start about 1 db to the right of the Arthurs and Dym curves. For large q-values the Akima curves fall excellently above and below the lower bound of the Arthurs and Dym curves. Lieberman (61) used the same mathematical model for the investigation of a closely related problem and Helstrom (60) give an approximate formula for the error probability of coherent MFSK systems which is identical with Arthurs and Dym's lower bound (6B).

Akima also has a family of error curves for an MFSK system with noncoherent detector. His equation is based on original work by Reiger (58) and Lieberman (61). Akima's curves for noncoherent MFSK appear as the utility area 5B in our fig. 12. Again we used both limits of our equation 2 when converting the results from digit error ratio to bit error ratio. It is significant that the difference of the utility of a coherent MFSK system as compared with a noncoherent MFSK system is only about 1 db. This value is practically independent of q, the number of frequencies.

Looking at all the ideal models of MFSK systems in fig. 12, one wonders if, at least theoretically, such systems could reach Shannon's upper bound (curve 1) when q is going to infinity. This is indeed the case. Turin (59) has shown that the utility approaches the value $-\log_e 2$ for $D \rightarrow 0$ in all orthogonal systems with optimum coherent detection, no matter what error ratio is desired. Thus it has been proved that the curves 5A and 6B will asymptotically approach curve 1. But to achieve this goal one has to use unrealistically high q values. Jacobs (63) showed that the same asymptotical behaviour holds for noncoherent MFSK systems ; Splitt (63) independently confirmed these results and showed that the same asymptotic approach also holds for multiple time shift keyed (MTSK) systems.

All these mathematical models assumed a coherent transmitter, a matched filter extractor (either for the complete waveform or the envelope waveform) and a maximum likelihood decider. No special waveform must be assumed, but a rectangular envelope and a constant frequency and phase throughout a digital interval is the logical choice. Waveform distortions and cross talk between the q frequencies are not considered. Ideal synchronization is assumed.

These idealizations never exist in practice. For binary coherent and noncoherent FSK systems, Stein (64) derived expressions for the error ratio under full consideration of the cross talk due to non-ideal filters and under the assumption of partially correlated noise components at the output of the filters. Stein's paper may serve as a starting point for the derivation of similarly improved mathematical models of MFSK systems. Any improvements of this type have to start with the detailed evaluation of the spectra of MFSK waveforms. The rectangular envelope of the digital signals is not necessarily the optimum waveshape ; it was merely a convenient starting point for the first models. Sunde (59) published a very detailed investigation of this problem of ideal pulse transmission, comparing the characteristics of AM and FM for this purpose. Although his paper is restricted to the binary case, his conclusions still apply to the individual waveforms of a MFSK system. These conclusions indicate that intersymbol interference can be avoided in FM-type transmissions without the need of a wider filter band than in double-sideband AM, when applying partial pulse shaping by premodulation and post-detection low-pass filters. This fundamental work of Sunde was further amplified by the previously mentioned research of Bennett and Rice (63a) that included the case of a discontinuous phase at the transitions between the digital elements. The papers by Salz and Stein (64b) and by Tjhung (64) discuss the spectra of telegraphic characters in FSK, the distribution of the instantaneous frequency for signal plus noise in q-ary FM, and the band occupancy of digital FM. A short report of Kobos (65) may offer some hints about codes for " quantized frequency transmission " (QFT), which is applied in military communications systems. As far as the transmission system is concerned, QFT is simply another name for MFSK.

Any attempt to improve the mathematical models of the receiving system must begin with the non-ideal character of the detection system itself. The models leading to the utility curves of fig. 12, Nos. 5 and 6, assume either ideal envelope detector or ideal coherent demodulator. The deficiencies of the first variety have been demonstrated excellently by Fubini and Johnson (48). The deficiencies of the second variety have only more recently come under detailed investigation. We showed in fig. 11 that for the binary receiver three different kinds of synchronous (or coherent) demodulators are applicable. The same applies to MFSK systems.

In the first case one applies for each signal frequency a separate local oscillator at the constant centre frequency of the respective bandpass filter (R_2 in fig. 10). These local oscillators must all be synchronized from a master timer.

For coherent reception a phase-lock loop is applied (R_5 in fig. 10). Many investigations have been made in the last

years of the performance of phase-lock loops, primarily for the demodulation of analog signals or for the acquisition of carrier frequencies in space communications systems (Filipowsky and Muehldorf 65a) or in tracking and navigation systems (Jaffee and Rechtin 55 ; Viterbi 63a, b ; Tausworthe 66).

G3. *MFSK Systems with Bandpass Extractors (Small Signal Base)*

These systems are modelled after the binary receiver arrangement R_1 or R_2 in fig. 10, whereby the noncoherent receiver R_1 is preferred. The receiver R_2 would need a separate oscillator for each frequency or a complex frequency synthesis system to replace individual oscillators. The mathematical models 5A and 5B show that the theoretical advantage to be bought with this considerable increase in circuit complexity is an improvement in utility of only 1 db. Channel instabilities (fading, phase distortions, frequency displacements) are likely to reduce this small advantage even further.

Jordan et al (55) operated an experimental MFSK system of this kind. They operated it in connection with a teletypewriter and obtained best results with a seven-frequency MFSK system. They used a pair of pulses to express each of the 32 teletypewriter characters, allowing each pulse to use one of the following seven tone frequencies : 1778, 1849, 1923, 2000, 2080, 2163, 2250 Hz. The system operated over voice channels occupying a bandwidth from 1740 to 2290 Hz, or a total of 550 Hz. It operated at the rate of a standard teletypewriter, which needs 153 ms for one character of 5 bits (a rate of 30.6 bps). The result is a bit density of $D = 0.057$ or -12.5 db. This operating point is entered as 4B in fig. 12, using the author's power contrast figures. As a controlled experiment the authors operated the same equipment in the binary mode and obtained results that are plotted as line 4A in fig. 12, with the upper end of the line representing the reported peak performance. All values are for a binary error ratio of 10^{-3} and for laboratory tests with white noise in the transmission band. During the tests, filter-shaping effects and the influence of simulated pulsive disturbances were investigated.

The question arises if there is an optimum number (q_{opt}) of frequency levels for achieving the highest bit density. Khanovich (64) investigated this problem and, for an operation with low energy contrast (relatively high error ratio), reaches the same conclusion that we indicated for the ideal systems : For a constant bandwidth, the transmission rate has a rather flat maximum around $q = 4$. Khanovich also shows that, for systems with higher energy contrast (smaller utility and smaller error ratio), this maximum shifts to higher q values, but always remains quite flat.

In section G6 we shall see that the MFSK systems are preferred to MPSK or MASK systems for operating over fading channels. In this connection it is important to have decision systems which can operate independently of the absolute output level. Thomas (60) reports an improved decision technique that is applicable to systems with bandpass extractors. Although Thomas describes his improvement in connection with a binary FSK system, it may be of interest to MFSK designers. Wards' (62) short note on the probability of error for " largest-of selection " could be helpful when transferring such practical or analytical results from binary to nonbinary receiving systems.

G4. *MFSK Systems with Matched Filter Extractors (Large Signal Base)*

The theoretical superiority of MFSK systems with matched filter extractors has been analytically demonstrated by Reiger (58). Recently Glenn (66) investigated wide-band MFSK systems where the signal base for each of the waveforms is larger than 10, while, in the models represented by curves 5 and 6 in fig. 12, a signal base of one had been assumed for the individual waveforms. We plotted in fig. 11 the values of Glenn's charts for a bit error ratio of 10^{-3} and a $\beta_S = 100$. As there are many more design difficulties for matched filters in a carrier band than for low-pass matched filters, Glenn uses a receiver arrangement where each MFSK subchannel has first a bandpass filter of the bandwidth of the waveform assigned to this subchannel. This is followed by an envelope detector and by the actual matched filter as a low-pass filter. The outputs of these filters in samples are compared in a maximum likelihood decider. If a system of this kind is to operate with high utility, the envelope detectors must operate at or below 0-db power contrast. Glenn discusses this problem of small signal detection in full detail.

The advantage of such wide-band MFSK systems is primarily in the area of anti-jam systems and security systems, as it is difficult to detect at all the presence of wideband signals that, at the receiver location, have a much smaller spectral density than the white noise. The detection problem of such systems is related to the problem of radar signal detection, which elicited much interest in the 1940s and is treated in a special monograph (Marcum and Swerling 60).

A system with the matched filters directly in a carrier band of 720 kHz bandwidth is the " Orthomatch " system of the Lincoln Laboratory of the MIT, Lexington, Massachusetts (Kuhn et al 63, Kuhn 64). Operating at a rate of 160 Kbps, its calculated and measured utility curves are shown as curves No. 9 in fig. 12 by a solid line and by a line of crosses, respectively. It is amazing how close these values approach the curves of the ideal system. The same laboratories designed another MFSK system as a military satellite communications system (Maguire 65). The anti-jamming feature of band-spread systems is again the principal reason that the system operates over a bandwidth of 20 MHz somewhere in the X-band (around 8 GHz). Known as the Lincoln Experimental Terminal (LET), it operates with the Lincoln Experimental Satellite (LES). LES-2 was launched May 6, 1965, into a synchronous orbit. The primary purpose of the system is, according to I. L. Lebow of the Lincoln Laboratory, " the communication of vocoded speech in a hostile environment " (Lebow 65). LET is designed to work with both active and passive satellite repeaters. Vocoded speech is speech converted to binary digits with the help of a device called VOCODER (voice coder and decoder). The output of the digital VOCODER is available in 9600 bps or 4800 bps data streams. The MFSK transmission system operates with $q = 16$, $\beta_S = 1$ and a digital rate of 5000 digits per second (dps). Thus the system can handle $5000 \,.\, \log_2 16 = 20{,}000$ bps, i.e., more than two times or four times the VOCODER output rate. This redundancy is used for a sequential coding and decoding system, which ensures practically error-free operation (Lebow 65). The jam-resistance of the system results from the fact that the 20-MHz transmission band is subdivided into 4096 subbands of about 5-KHz bandwidths each. Only 16 of these bands are selected at any one time for use as the 16 MFSK subchannels. The selection rules are known to all participating stations and are changed frequently.

From the available data and with the help of a few assumptions by this author, the utility lines for this system may be placed approximately at the positions of lines 10A and 10B in fig. 12, assigning as total bandwidth merely the bandwidth of 16 subchannels and not of 4096 subchannels. Line 10A corresponds to the 4800-bps rate and line 10B to 9600-bps rate.

The characteristics that make matched filter MFSK systems jam-resistive are also desirable characteristics when operating over heavily dispersive media, such as the use of an orbiting dipole belt (project Westford) for communications (Lebow et al 64). This kind of application will be discussed further in subsection G6.

G5. *Digital FM Systems*

The examples of binary FSK systems in fig. 10 showed combinations of T_3 with R_4, R_5 or R_6, which are essentially analog FM systems transmitting a binary waveform and adding a slicer at the output. The slicer can be replaced by a clocked resampler and a decider to restore the accurate binary elementary length of each element. When feeding this system with a multilevel low-pass waveform (since it is the result of a video MASK system), the total system becomes a special variety of an MFSK system which best may be designated as MASK-FM (section F2).

In fig. 11 we showed that the systems of this class operate at the right side of the utility chart, similar to the MASK systems. Their advantage over MASK systems is their insensitivity to variations in attenuation of the channel (fading) or gain of the amplifiers. To compete in bit density with MASK systems, digital FM systems are required to operate as small-band FM systems.

An experimental digital FM modem (modulator-demodulator set, or data terminal) was developed in the Bell Telephone Laboratories and described by Salz and Koll (66). We plotted the results of Salz and Koll's laboratory tests with line simulators and noise generators in fig. 12, area 8. The operating point 8A is for binary operation; 8B, for duobinary operation (duobinary MASK signals transmitted by FM); and 8C, for quaternary operation (four-level MASK transmitted by FM). The line from 8A to 8C represents the change from $q = 2$ to $q = 4$.

A number of related investigations should be mentioned to assist development engineers who intend to explore further this interesting field of digital FM (or quantized FM, as it is frequently called). Lender's (65b) paper on binary orthogonal FM techniques was mentioned in section G1. It will be helpful to clarify transmitter and receiver problems for MASK-FM. Bennett and Salz (63b) disclosed the original analysis underlying the development of the BTL digital FM terminal discussed previously. Of particular importance is the theory of digital FM spectra. Distortions, transmission bandwidth, and peak deviation are all key parameters in determining the bit density of digital FM systems. Sunde's 1959 paper became a standard work, which later was expanded by Bennett and Rice (63a) to an analysis of the spectral density and autocorrelation functions associated with binary FSK. Building on this analytical background, Anderson and Salz (65b) investigated the spectra of digital FM of four and eight-level signals, and Salz (65a) derived compact formulas for the spectral density function of an ensemble of continuous-phase constant-envelope multilevel FM waves. Independently of the above authors, Pelchat (64) arrived at an expression for the power spectrum and the autocorrelation function of PCM-FM waves.

An interesting possibility for digital FM detection has been analyzed by Anderson et al (65) for the binary case. Called the differential detection method, it involves the detection of binary FM by multiplication of the received signal by itself, delayed for one elementary length.

From Germany comes a report of laboratory tests of a four-level digital FM set (Von Hänisch and Kettel 63). The results yield the operating point 11 in fig. 12 for the four-level set. This is close to the binary FM operating point of the BTL experimental digital FM terminal, while the corresponding binary operating point of the German set is about 2 db lower and 2 db to the left.

G6. *Special Applications of MFSK*

FSK systems are generally praised for their relatively good performance under *atmospheric or pulsive noise* disturbances. Bello (65b) compared an FSK modem with 16 subchannels operating in parallel with a quaternary PSK modem, showing a slight advantage (about 3 db) in utility for the FSK system under HF radio transmission conditions. Mrs. Conda (65) analyzed the behaviour of a noncoherent FSK system (NCFSK) under atmospheric noise conditions. She provides interesting design charts that should be of great assistance to systems planners of such links. The effect of impulse noise, like that encountered in telephone voice circuits, on FSK systems is the subject of two investigations. Bodonyi (61) found that noncoherent FSK systems yield the lowest error ratio of all binary systems compared under the influence of heavy impulsive disturbances. Engel (65), however, gave a different rating, finding in the presence of impulsive noise no different ranking of the competitive digital transmission methods than under Gaussian noise conditions. His results leave the FSK system as the second worst, being inferior to SSB, DSB, and PSK, and being superior only to AM with envelope detection. He used a binary FM system with frequency discriminator.

Although the previously mentioned tests applied to binary FSK systems, it is evident that their results are highly indicative of the behaviour of MFSK systems under pulsive noise conditions. More direct investigations of the performance of MFSK are available in connection with the transmission over *fading channels* and particularly in connection with diversity systems.

Jordan (61) compared the various *anti-multipath* methods known at that time. He concludes that multilevel FSK will require, under certain circumstances, less penalty in transmission rate for anti-multipath measures than binary FSK. Servinskiy (64), Dumanian (63), and J. N. Pierce (66) offer curves that may help in the design of "parallel FSK channels" (MFSK) to find the best compromise between complexity, power, and bandwidth when the digital length T_D is prescribed because of the encountered multipath conditions.

The advantages of *diversity reception* have been known for some time (Van Wambeck and Ross 51). J. N. Pierce (58) analyzed methods of using diversity reception to improve FSK systems when operating over Rayleigh fading channels. The same author (J. N. Pierce 61) amplified his investigations by a note comparing space diversity with frequency and time diversity when transmitting binary signals. The theoretical diversity improvement in MFSK systems is the subject of an analysis by Hahn (62). He concludes that MFSK systems can be improved by diversity methods when operating in the presence of random noise and fading. He recommends, as the optimum combining method, that one square and add the detected outputs of corresponding filters from each diversity channel and then apply the maximum likelihood decider.

Kennedy and Lebow (64) gave an introduction to the principles of communications over dispersive channels. They developed a channel model, presented simple formulas and curves for the selection of communications parameters for operation over multipath channels, and discussed binary FSK and MFSK. An application of these fundamentals to the orbitting dipole belt channel, a highly dispersive channel resulting from project "Westford", is discussed by Lebow et al (64) and Brookner (65b).

The application of MFSK systems as anti-jamming systems or secure communications systems was discussed in subsection G4. Here we may mention Culbertson and Perkins (64), who described a solid state digital-to-digital converter for use with high frequency SSB transceivers, utilizing the principle of quantized frequency modulation (QFM). An airborne flight test model was used in 1963 in test flights, comparing this system with an audio frequency shift keying (AFSK) system. Another interesting application of MFSK is a random-access communications system using frequency shifted pseudo-noise (PN) signals, described by Blasbalg et al (64).

Somewhat related to MFSK, but better classified as polysignal systems, is a digital data transmission model originally described by Renshaw (64). Continuing these studies, Renshaw (65b) described a signal processing system and a delay line time compressor (DELTIC), which can be used

in connection with the multitone parallel FSK system mentioned above. This combination, according to the author, allows much larger signalling alphabets without paying the penalty of excessive equipment complexity.

Another piece of related equipment is an experimental terminal of the Bell Telephone Laboratories (R. C. Anderson et al 65). Although it falls into the class of polysignal systems, it is mentioned here because it was used in tests over a cross section of the various types of facility found in the switched network. The set transmits data in binary FSK simultaneously over nine subbands within the voice band. Relationships between error performance and transmission characteristics were investigated.

As a last application of MFSK techniques, the multifrequency signalling system of the dial telephone service should be mentioned. The receiving part has much in common with MFSK receivers (Pommerening 65). The transmission experience over the switched network and the error rates of such dial signals should have significance for the design of MFSK modems for voice circuits.

In conclusion, it should be noted that developments are progressing on the large scale that should be expected when looking at fig. 11, which shows the range of MFSK systems extending over all decades of bit density.

H. Multiphase Shift Keying (MPSK) Systems

H1. *Fundamentals of MPSK Systems*

The oldest application of phase modulation for digital transmission is most likely a kind of tone telegraphy which operates in a bipolar two-state mode as a double sideband suppressed carrier (DSB-SC) system. In this mode a Mark corresponds to one phase (or positive amplitude), and a Space corresponds to the opposite phase (180-degree shifted or negative amplitude) according to the identity :

$$-\sin w_c t = \sin (w_c t + \pi) . \qquad (42)$$

This DSB-SC or biphase system is also the prototype for the ideal binary system shown as curve 3 in all utility charts (figs. 8 or 12). Montgomery (54) established the theoretical superiority of such a biphase system over FSK and ASK for both cases—no fading or Rayleigh fading of the carrier.

Doelz (53) suggested a four-phase system which he published later in more detail (Doelz et al 57). Since then it has become known as the Kineplex system (subsection H3). Losee (58) compared the binary and quaternary digital phase modulation techniques. His paper reflects the opinion of data transmission equipment designers on PSK at that time. Since then PSK has gained the attention of many laboratories and its performance is well understood, both analytically and practically.

PSK is credited with combining the advantages of both ASK and FSK, without suffering from the shortcomings of these systems. Although this statement is a generalization as the result of oversimplified theories, it cannot be denied that PSK requires less bandwidth than FSK and offers advantages over ASK when used over slowly time-varying channels. The greatest problem with PSK is the requirement for a reliable phase reference in the receiving set. The many varieties of PSK systems differ in the methods used to establish this phase reference. There are two major classes of PSK and MPSK systems. The first, called the automatic phase reference (APR) class, transmits the phase reference in the same signals that carry the information (subsection H3). The second, called the separate phase reference (SPR) class, provides the reference by means that differ from the information-carrying signals (subsection H4). However, a discussion of the fundamentals of the binary PSK systems will help to understand the details of the MPSK systems, described later.

Figure 13 shows 12 frequently used modes of carrying binary signals in a low-pass channel (video channel or base

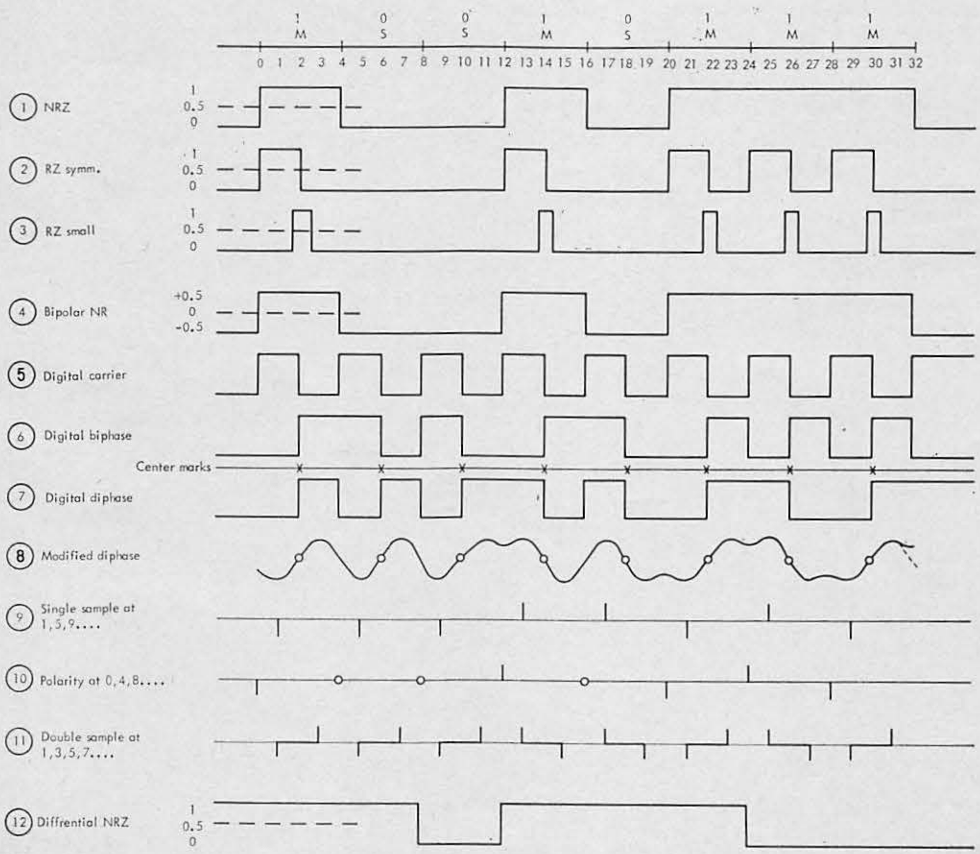


Figure 13.—Binary information formats.

band channel). Waveform 1 is the most usual two-state mode with Space corresponding to one voltage level (0) and Mark corresponding to another voltage level (+1). If several Marks follow each other, the voltage does not return to zero (NRZ) (Bennett and Davey 65). This waveform apparently has a dc component that is the average voltage of the waveform. When discontinuing transmission, the waveform will stop in either state. This is not the case with the return-to-zero (RZ) waveforms 2 and 3, although these waveforms have a dc component. A bipolar non-return waveform (No. 4) could avoid a dc component in the special case where there is an equal number of Mark and Space symbols.

The suppression of a dc component is possible through the introduction of a rectangular subcarrier of a fundamental frequency exactly equal to the bit rate. The information symbols modulate this subcarrier in a biphase mode. Waveform 5 is the digital subcarrier. The resulting waveform (No. 6) can never stay longer than one bit length in the same state ; i.e., no dc component can materialize.

The digital diphase (7) is the differential equivalent of the digital biphase. It always takes the phase during the previous symbol and reference and, if the input information element is a Mark, changes the phase ; if it is a Space, it retains the same phase.

Naturally, a waveform with rectangular elements has an unnecessarily wide spectrum. Therefore, RCA introduced a filtered version of the diphase waveform, which it calls " modified diphase " (Douglas et al 61). This waveform is shown as No. 8 in fig. 13, where it has zero crossings at the centre of each bit interval. This characteristic is very helpful for carrier extraction. Line 9 in fig. 13 shows that a resampling operation at the first quarter of each interval (i.e., at the instants marked 1, 5, 9, . . .) at the top of fig. 13 is a good way to recover the information at the receiver. Line 10 shows that the modified diphase waveform carries even more information about the received message. When resampling at the separation lines between any two message elements (i.e., at the instants 0, 4, 8, . . .), the receiver may detect the absolute polarity of the elements without any reference to the previous element. If the sample is zero at this instant, a Space is the message element carried by the following interval. If a positive or negative pulse is the result of the resampling process, a Mark is carried by the following element. Line 11 finally shows another form of redundancy and its utilization. It has been called " double sampling detection " (Atzenbeck 65). Each message interval is now resampled twice, at the one-quarter point and at the three-quarters point (i.e., at the instants 1, 3, 5, 7, . . .). If a pair has the same sequence of polarity as the previous pair, a Space has been received ; if the sequence of polarities of the pair of pulses differs from that of the previous pair, a Mark has been received. If the two pulses of a pair have the same polarity, a mutilated element has been received, as this is a " forbidden " combination for a pair of resamples. Error-correction schemes or ARQ (repetitious) operations can then act to avoid an output error.

Line 12 shows another mode of encoding that is used occasionally. It is designated as differential no-return-to-zero (DNRZ) and it follows the rule that a " zero " level is produced when the element has the same symbol as the previous element, and a " one " level is produced if it has the opposite symbol. The weakness of this waveform is obvious when compared with digital diphase.

After this review of binary formats, it will be easy to survey the many transmitter and receiver arrangements that have been devised for binary PSK. They are shown in fig. 14 in a presentation similar to the binary FSK arrangements in fig. 10. MPSK systems evolved from these fundamental binary arrangements. Basically, the various modes of carrier extraction are the same for PSK and for MPSK.

There are four transmitter arrangements and nine receiver arrangements. As in the FSK case, there are some combinations more efficient than others. They are indicated by shaded areas in the central matrix.

The transmitter arrangement T_1 is the one that many of the early PSK systems used. It has as an optional feature the generation of a pilot frequency f_p (Walker 65).

The transmitter arrangement T_2 is applied in those cases where the carrier reference information is transmitted via

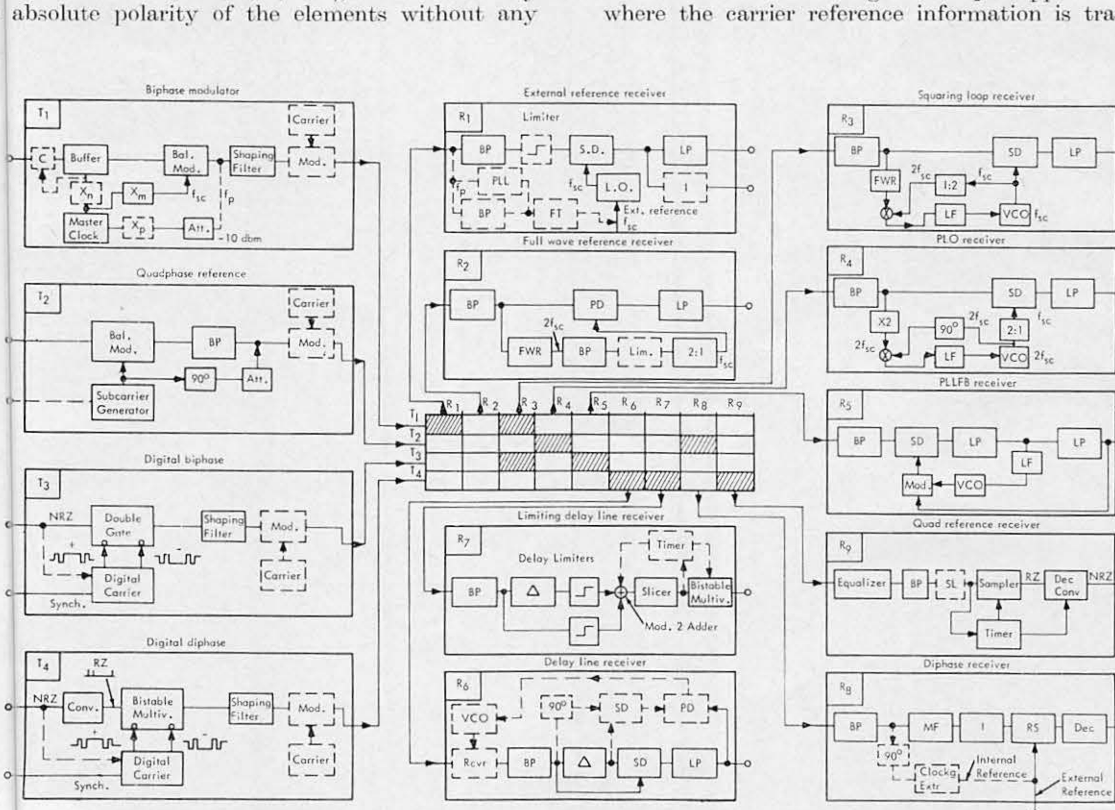


Figure 14.—Circuit arrangements of binary PSK systems.

the quadrature component of the subcarrier. An important design problem is the determination of the fraction of the total transmitter power which should be assigned to the synchronization channel. The adjustment of this power percentage can be made with the help of an attenuator (ATT in fig. 14, T_1 and T_2) (Stiffler 64b).

The arrangement T_3 is used to generate the previously mentioned digital biphase signals ; and T_4, the diphase signals.

The receiver arrangements in fig. 14 show nine varieties, each of which differs in the methods applied to recover the carrier phase and to establish bit synchronization.

The arrangement R_1, well known from radar and ranging systems, is applied when an external local reference is available. The rather complex synchronization circuits used to achieve bit synchronization are not shown in fig. 14, R_1, but a good presentation of these problems may be found in Hopner (59).

The receiver arrangement R_2 is based on the fact that it is easier to recover the doubled carrier frequency than the carrier frequency itself. Because of the biphase operation, the information is consistently in phase with a frequency equal to twice the carrier frequency, although it is partly in phase and partly out of phase with the carrier frequency itself. The difficulty with this arrangement is that special measures are needed to determine the absolute polarity, which, naturally, is lost in the full-wave rectification process (Fisch 63, Critchlow et al 64).

The arrangement R_3 is generally known as the squaring loop (Lindsey 66, Stiffler 64a). It combines the advantages of the full-wave rectifier with those of a phase-lock loop for carrier frequency recovery ; however, it also requires external means to determine the absolute polarity of the received signals.

R_4 is a slightly modified version of the squaring loop which operates the voltage-controlled oscillator (VCO) at twice the carrier frequency (Hannan and Olson 61), whereas, in the arrangement R_3, the VCO operates at the carrier frequency.

The arrangement R_5 shows a phase-lock loop with feedback (PLLFB), where an attempt is made to assure the correct polarity by remodulating the VCO output with the recovered information, to perform in the synchronous demodulator (SD) a complete correlation process of the noisy waveforms with a clean, locally generated waveform (Choate 60).

R_6 shows the typical receiver arrangement for binary phase differential reception. It requires a delay element to store the phase of each elementary interval during one interval length for comparing it with the next element. Glenn (60) compared this arrangement with other binary transmission methods and pointed out the disadvantage that the reference signal itself depends on the input signal level. Thus it can be demonstrated that this receiver arrangement breaks down faster, at low signal levels, than other similar arrangements. It has been used in a radio modem for shortwave reception of biphase signals. Kolarcik and Paramithas (63) compared its performance with that of an integrate-and-dump arrangement as used in Kineplex (subsection H3). It is also used in an AFC modem as a phase-correlation system (Losee 58).

The receiving arrangement R_7, known as the Swift receiver (Clark 64), evolved from R_6 by introducing limiters into the delayed branch and the direct branch.

The arrangement R_8 is used in a subchannel modem of a successful radio data transmission system (Kathryn) (Bello 65). It operates with a matched filter extractor and with an integrate-and-dump circuit, similar to the quaternary Kineplex system (Doelz et al 57).

R_9 is the typical receiving arrangement for the previously discussed diphase system (transmitter T_4) (Douglas et al 61, Atzenbeck 65, Kolarcik and Paramithas 63).

Concluding this review of biphase circuitry, it should be stressed that no all-inclusive comparison of these varieties of arrangements seems to be available in the literature. This is only natural, since the requirements of various applications are quite different and since many of the circuit innovations are still under intensive analytical exploration.

Fundamental considerations for the comparison of phase shift keyed data transmission systems may be found in the papers by the Armour Research Foundation (61b), Khvorostenko (64) and Bussgang and Leiter (66).

H2. *Ideal MPSK Systems and Practical Tests*

Two major classes of MPSK systems were mentioned—the automatic phase reference system, and the separate phase reference (SPR) system. In this subsection we present the ideal mathematical models for both classes, beginning with SPR.

We again show a utility chart in fig. 15, which contains, for reference purposes : curve 1, representing the ideal Shannon system discussed in Part I ; curve 2, displaying the utility curve of the ideal MASK system discussed in subsection F1 in this part ; and curve 3, the ideal binary system with $\lambda = -1$ and with matched filter extractor, which demonstrates the upper limit of utility for any conceivable uncoded binary system with bit-by-bit decisions.

Arthurs and Dym (62) developed an upper and lower bound for the error probability of an ideal MPSK system with external reference (coherent MPSK), assuming again an optimum extractor (matched filter plus synchronous detector) and an optimum decider. We converted these bounds into bounds for the utility, arriving in our notation at the equation :

$$\frac{1}{\log_2 q} Q\left\{\frac{2\log_2 q}{\sqrt{u}} . \sin\frac{\pi}{q}\right\} \leqslant \bar{e}_B \leqslant \frac{q-1}{\log_2 q} Q\left\{\frac{2\log_2 q}{\sqrt{u}} . \sin\frac{\pi}{q}\right\} . \quad (43)$$

The error function Q (x) has been defined and plotted in fig. 8b. Equation 43 is similar in its form to equation 39 for MFSK. The argument of the error function, however, is different, as is the equation for the bit density, which now (for MPSK) is simply as follows :

$$D = \frac{\log_2 q}{\beta_D} . \quad (44)$$

In both equations q is the number of phase states used in the system. The states are assumed to be equally distributed over $2\pi = 360°$. β_D is the signal base for one digit. The waveforms of all digits are assumed to have rectangular envelopes and constant phase over the whole digital duration. It can be seen that β_D does not enter equation 43. This assumes the use of the minimum signal base that is compatible with distortion-free operation. For ideal rectangular waveshape, this signal base would be much larger than one. On the other hand, there is the assumption of a matched filter extractor that would restrict the noise bandwidth of a rectangular signal to about $B_D = 1/T_D$. Thus it seems that for an ideal mathematical model, $\beta_D = 1$ is a reasonable assumption. Efforts to develop more advanced mathematical models considering spectral limitation in detail will be reviewed below. We plotted the utility area resulting from the Arthurs and Dym's model as area 4 in fig. 15 for $\bar{e}_B = 10^{-3}$ and $\beta_D = 1$.

It is not surprising that, under the given assumption, both curves start for $q = 2$ at the value $u = -6.8$ db, which is identical with the value for the ideal binary system. Indeed, for $q = 2$, equation 43 becomes identical with equation 34. The upper limit of the utility reaches a flat maximum close to $q = 3$ and then decreases, generally running parallel to the ideal Shannon model (curve 1). This maximum at $q = 3$ has been pointed out by Khanovich and Bondarev (65) and also by many other authors. It is possibly the paper by Cahn (59) which shows this maximum for the first time in fig. 4 of his paper. There it appears as a minimum,

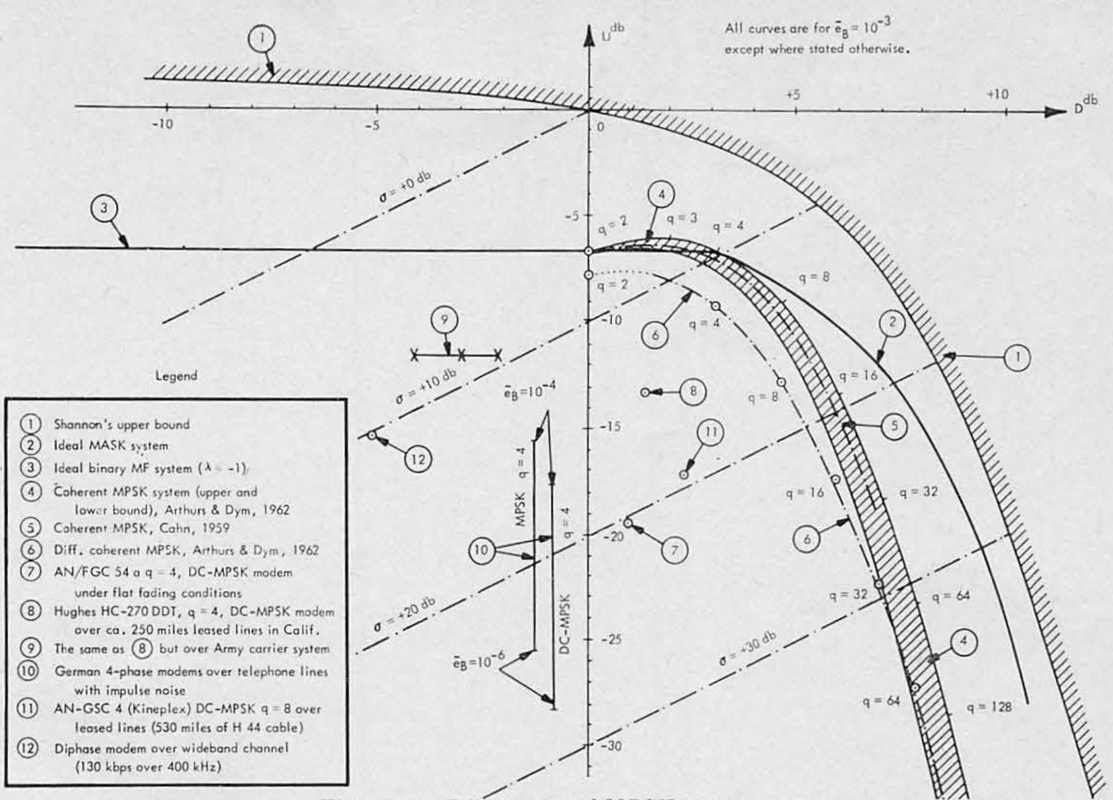


Figure 15.—Utility curves of MPSK systems.

due to the different kind of normalization. Cahn's curve of the coherent MPSK system, as plotted in our notation, falls exactly between the upper and lower bound of the Arthurs and Dym model (curve 5, fig. 15). Notice that Cahn does not use a matched filter extractor but resamples the phase at the peaks of the received waveform. Yet, for $\beta_D = 1$, both methods show the same extraction efficiency, because the efficiency of the MF extractor is proportional to the TB product (β_D) of the signals. Again, one must remember that these models are based on rather ideal assumptions and that neither system will be free of intersymbol interference when operating with $\beta_D = 1$. Practical systems will show a difference between the two cases.

Mathematical models are also available for the other class of MPSK systems, the APR class, i.e., the systems with automatic phase reference. Although the noisy signal is compared with a clean reference in the SPR mode, it is evident that in the APR mode two noisy signals are compared with each other. We may say, in a crude fashion, that there is twice as much noise present and therefore a 3-db degradation is to be expected. It will be seen that this is, indeed, the case for $q > 4$.

Most APR designs merely use the phase difference between a pair of digits transmitted immediately one after the other, although other designs involving more digits are conceivable. This limit to the phase difference between two digits gave the systems of this class the name *differentially coherent*. Arthurs and Dym (62) gave an approximate formula for the APR case (equation 82, p. 353 in their paper). In our notation this can be written as follows :

$$\bar{e}_B \doteq \frac{2}{\sqrt{\log_2 q}} \cdot Q\left\{\frac{2 \cdot \log_2 q}{u} \cdot \sin \frac{\pi}{q \cdot \sqrt{2}}\right\}. \tag{45}$$

The bit density for this model is given by equation 44, as in the previous case. Equation 45 is plotted as curve 6 for $\bar{e}_B = 10^{-3}$ and for q values from 4 to 64. It is not accurate enough for $q = 2$. For this extreme case (binary differentially coherent PSK), Arthurs and Dym gave an explicit formula which, in our notation, can be written directly in decibels as follows :

$$u^{db} = -10 \log_{10} (-\log_e 2\bar{e}_B) . \tag{46}$$

Comparing equation 46 with equation 38a of subsection G2 shows that the ideal binary differentially coherent PSK (DC-PSK) system is 3-db better than the ideal noncoherent FSK system. That is true for all error ratios ($\bar{e}_B$) and has been stressed by many authors (Cahn 59, Lawton 58). Salz (65b) compared the ideal differentially phase-coherent system with the previously discussed digital FM systems and reached the conclusion that, for large q values, both systems have nearly the same utility. For $q = 64$, Salz found that digital FM is only 0.712 db below DC-PSK. The error statistics of DC-PSK systems are important. It is reasonable to assume that errors may occur in pairs because an error in one digit will also base the next decision on an unreliable reference. Salz and Saltzberg (64) analyzed this phenomenon and showed, for additive Gaussian noise, that the ratio of double errors to single errors is less than 0.25 when the SNR is small, and that the ratio approaches zero for large SNR. Experimental tests verified this result but also showed that most errors occur in isolated pairs when isolated noise pulses are the principal disturbance. Bussgang and Leiter (63) provided another significant improvement of the theory of DC-PSK systems. They used decision theory to improve the accuracy of Arthurs and Dym's equation 81 in the range where it is most inaccurate ($q \leqslant 4$). They arrived at an exact expression for $q = 4$.

Another assumption of the ideal systems which is subject to improvement is the width of the decision threshold. The original theories reviewed above assumed that the decision threshold will be placed at the optimum level and that the maximum likelihood decision (MLD) rule will be applied. This includes the assumption that an infinitely sharp threshold is available. Practical circuits cannot provide such an ideal situation. Hubbard's (66) paper will therefore be welcomed by those who attempt to find more realistic approaches to mathematical models. He analyzed the effect of a finite-width decision threshold on binary DC-PSK systems. This analysis may be extended to MPSK systems. Another significant step towards realisti-

cal mathematical models is the paper by Becker et al (64), which analyzes the influence of frequency offset on DC-MPSK systems. Lindsey (65b) extended the analysis of PSK systems to include the effects of timing errors on the detector output statistics. He indicated the deleterious effects of noisy phase references by graphical representations. This may become a very important aspect of MPSK theory.

Interesting contributions to the theory of MPSK systems came from Bussgang and Leiter (64) and from Proakis et al (64). These papers deal with test statistics for an optimum DC-MPSK system. Leiter and Proakis (64) showed in a short note that the test statistics of the above papers were actually equivalent.

Despite all these improvements by the above-mentioned contributions, there is one major deficiency in the mathematical models : the consideration of the bandwidth occupancy. In only a few applications does it not matter how far the sidebands of a MPSK wave range into other, neighbouring frequency bands. In most applications there is a very sharp band limitation, which has to be enforced by international agencies, military frequency control personnel, or simply by common sense of all the users of a wider frequency band. In any case it is clear that bandwidth conservation (or spectrum conservation) is a major objective in an age of tremendously fast expanding communications facilities. Yet it is only very recently that the engineering community is turning its interest to this vital problem. MPSK systems are particularly vulnerable to the accusation that they unnecessarily occupy wide frequency bands, because the mathematical models of the systems assume infinitely steep phase transitions at the end of each waveform. MFSK systems are better in this respect since they attempt to operate the transmitters with coherent waveforms ; i.e., the transition from one frequency to another is established with continuous phase. There is no discontinuity in the amplitude at the transition between two digital elements. By definition, this can not be the case with PSK systems, though the enforcing of an integer ratio between subcarrier frequency and digital transmission rate can help to avoid excessive discontinuities. In any case one must recognize that the spectral distribution of the energy of MPSK waves is a major problem, deserving careful analytical and experimental investigation. In this sense it is gratifying to review the progress over the last few years.

Sunde (61) directed the attention of designers to the problem of the spectral distribution when comparing different modulation methods and their susceptibility to phase distortions in the transmission channel. He pointed out that, although PSK has an advantage of a few decibels in utility over FSK under ideal conditions, this advantage will fast turn into inferior performance when phase distortions (delay distortions) affect the channel. This and similar statements of other authors led to a number of investigations of the spectral distribution of phase shift keyed signals. Rumble (64) compared the spectra of binary PSK systems under the various coding formats presented in fig. 13. Pushman (63) plotted the spectra of various binary modulation systems, including PSK for random messages ; and Karshin (64) discussed the influence of the IF bandwidth on binary PSK signals for various transmission rates. Karshin (65) points to the strong influence of the TB product (β_D value) of the individual PSK waveforms on the error probability of binary PSK systems. Petrovich and Razmakhnin (65) compared spectra of MASK and MPSK signals and concluded " that the phase keying spectrum remains practically the same as the number of phase gradings increases, and it is, in this sense analogous to the spectrum of amplitude-keyed waveforms ". Digital computer simulation (Rappeport 65) turned out to be a valuable tool when investigating the spectral spread of quaternary MPSK signals, and also their susceptibility to phase distortions and to impulse noise. Other closely related contributions come from Weber (65) and Pelchat (65). The currently available practical systems are compared in the utility chart of fig. 15 with the older, rather simplified mathematical models.

Point 7 of fig. 15 refers to the average results of a large number of field tests with U.S. Army equipment (AN-FGC 54) employing quaternary differentially coherent MPSK under flat fading conditions (Bello 65b). The equipment operates with 20 subchannels over a voice band. The subchannels are spaced 110 Hz apart and the signals are simultaneously keyed and mutually orthogonal (Kineplex type). The total bandwidth occupancy is 2400 Hz ; the transmission rate is 20×150 bps, giving the system a bit density of 1.25 or $+0.96$ db. The measured SNR is 20.5 db, yielding a utility of -19.54 db. It can be seen that this point is about 10 db below the theoretical point ($q = 4$) on curve 6. This degradation is primarily due to atmospheric noise and flat fading. Operational periods with multipath and Doppler effects had been excluded from the error counts. Diversity reception had been applied, although no details could be found in the paper about the particular diversity arrangement used. The paper, however, gives a valuable analysis of diversity operation for binary PSK systems.

Point 8 in fig. 15 reflects the average results of field tests of a quarternary DC-MPSK modem over leased telephone lines of approximately 250 miles in length (Evans et al 61). This modem operated with 2500 bps over an effective channel bandwidth of 1800 Hz. It has a single channel centred at 1667 Hz. It can be seen that in this case the operational equipment is only 4 db below the theoretical value and that the bit density also comes closer to its theoretical goal than in the case of point 7.

Line 9 is from another report about tests of the Hughes Aircraft Company (HC 270) four-phase modem over Army carrier systems (Tucker and Duffy 63). The system operated for these tests with 1200 bps. The first test was back to back with only the filters in the terminal equipment restricting the bandwidth (3000-Hz noise bandwidth). The equipment needed 7.2 db SNR to run with 10^{-3} error ratio. This corresponds to the left end of the line. The second test was over six sections of carrier equipment with all repeaters switched in, but no real lines connected. In such tests noise may be inserted from a noise generator if the natural noise of the equipment is not adequate. The paper does not specify in detail the kind of noise generation and noise measurement used during these tests. The purpose was primarily to check the operation under phase distortions and band limitations as they are inserted by the carrier equipment. The results of this second test fall into the centre part of line 9 in fig. 15. The right end of the line corresponds to a test where 12 sections of AN/TCC7 carrier equipment were connected in tandem. The fact that the utility remains constant over all tests indicates that the change in SNR ratio was due only to the bandwidth decreasing with the increasing number of sections in tandem. This is an interesting demonstration of the advantage of utility curves. Comparing these results with the theoretical curve shows that the military carrier equipment enforces an operation at much lower bit densities than those theoretically possible or even those achieved in radio transmission.

Lines 10A and B are the results of tests performed over German telephone facilities with controlled impulse noise intentionally inserted (Von Hänisch and Kettel 63). The results are available only for the range of error ratios from 10^{-4} to 10^{-6}. When comparing these results with others, it is necessary to extrapolate the lines upwards to about -12 db and -15 db, respectively, to estimate the utility for $\bar{e}_B = 10^{-3}$.

Point 11 is reported by Morioka et al (60). It resulted from tests of an octonary DC-MPSK system (AN-GSC4, Kineplex) operating over specially selected leased lines (530 miles of H 44 cable) at 5400 bps. The telephone circuits have a nominal bandwidth of 4 kHz, but 3200 Hz were used for the

signals. This gives a bit density of 1.69 or 2.28 db. The systems needed a SNR of 19.5 db for 10^{-3} bit error ratio. This established the utility at -17.2 db. The authors mention that " The impulse noise must not be too objectionable, although the operation of the equipment in the presence of impulse noise had indicated that it is not as susceptible as one might believe." The equipment operates with six subchannels centred from 935 Hz to 3135 Hz with 440 Hz separation. The digital rate in each subchannel is 300 bps.

Point 12 in fig. 15 finally is inserted for comparison (Langley and Oliver 65). It refers to the previously mentioned diphase binary PSK system when used for space booster telemetry over a wide-band channel of 400 kHz bandwidth. In this particular version the system operates in an error control feedback mode, which makes it practically error free, but which shifted its bit density to -5.12 db.

H3. *Auto-Phase Reference MPSK Systems*

Two or more subclasses of APR systems may be discriminated according to the mode used to give the receiver the phase reference and to update it continuously. Of all the conceivable means that can be used to achieve this goal, the differentially coherent (DC) mode is most frequently applied. Russian authors speak of relative phase telegraphy (RPT) and Bobrov (60) established an intricate classification system of the various versions which can be applied to transmit the phase reference together with the information. Basically there are two groups of RPT systems.

The first group covers systems in which the number of possible phases of the signal is equal to the number of phase differences used for transmission. The minimum phase shift is zero, meaning that one of the waveforms is exactly repeated with the same phase state to express one of the characters. This first group includes the binary PSK system with 0 to 180 degrees, the ternary PSK system with 0 to 120 to 240 degrees, and the standard DC-MPSK system with 0 to 90 to 180 to 270 degrees. The disadvantage of systems of this group is the unchanged phase state which results for certain messages. This may cause difficulties in circuits and may present problems for the synchronization extractor (recovery of the clocking rate).

A special encoding mode, used in a second group of RPT systems, and called the rotating phase mode (Nazarov 64) avoids this difficulty by providing two interlaced sets of waveforms offset in phase against each other for π/q and using them alternately. In other words, the transmitter will shift in a binary system for 90 to 270 degrees ; in a ternary system, for 60 to 180 to 300 degrees ; and, in a quaternary system, for 45 to 135 to 225 to 315 degrees. The last mode is the normal operation of the Kineplex system (Icenbice 57).

Stein (64) performed a unified analysis of binary communications systems, arriving at a fundamental mathematical formulation that should be of importance to further research into the noisy reference problem of APR-MPSK systems. The next step in the development of APR methods may well be the inclusion of more than just the last received signal into the process of recovering the phase references. Good guidance for such developments may be derived from Proakis et al (64).

There are not many MPSK terminals in existence. Therefore, we prefer to list them by developers and mention their characteristic features.

The *Kineplex system*, originally called predicted wave signalling (PWS) system, has already been mentioned several times (Doelz et al 57, Morioka et al 60, Icenbice 57). It operates with two, four, or eight phases and prefers the rotational phase encoding mode. In its standard version it operates with many subchannels (up to 20), which are all keyed simultaneously and whose subcarrier frequencies are so assigned that there is a minimum of intersymbol interference for a given elementary length of the rectangular signals (Mosier 57). Gated reasonators (infinite Q-filters) exactly tuned to the subcarrier frequency act as matched filters and as phase storage devices at the receiving side. Two sets of reasonators are alternatively used and quenched after two elementary intervals. They are loaded during the first interval of each circle and they hold the phase reference during the second interval. Kineplex has now been in development and operation for more than 10 years. Many evaluations have been performed in comparison with other systems in laboratory tests (Easterling et al 61), over telephone lines (Fontaine and Gallager 61, Fontaine 63) and over radio channels (Greim et al 65). The result may be summarized by the statement that DC-MPSK systems have absolutely fulfilled the expectations when tested under the conditions assumed by theory, i.e., Gaussian noise, linear, and time-constant channels. But when subjected to real life operational conditions, the results show degradation of 10 db and more against the unrealistically ideal mathematical models.

One four-phase *data modem of Hughes Aircraft Company* (HC 270) operates over a single subcarrier frequency located at the centre of a voice band (Evans et al 61). It is, therefore, less costly than the Kineplex terminals, but is using the available bandwidth less efficiently (no orthogonal multisignal composite waveforms) and is more subject to phase distortions, since the spectrum of each signal ranges over the whole voice band. It can operate with a rate of 600, 1200, or 2400 bps over telephone lines of the switched network and with 4800 bps over higher quality voice circuits. It applies an auto-correlation technique to identify the change in phase (Toffler and Buterbaugh 61), and it incorporates the design of a phase-locked loop (PLL), and an advance-retard counter to recover timing information from the received signal (Buterbaugh 62).

Another four-phase data modem of *Hughes Aircraft Company* (HC 273) was specifically designed for high frequency (i.e., short wave) radio transmission and operates with 12 subchannels, each using a digit length of 10 ms (200 bps) so that the total rate is 2400 bps. The digital length of 10 ms is sufficient to obtain satisfactory long-haul and short-haul operation over a wide range of frequencies.

A special differentially *coherent quadriphase modem* developed by *Republican Aviation Corporation* operates on a single carrier over a wideband microwave channel at 13.5 GHz (Fisch 63). It was used to demonstrate the transmission of digitized television signals digitally encoded at 24 megabits per second (mbps). The quadriphase modulation is straightforward in the phases 0, 90, 180 and 360 degrees. The inclusion of 0-degree phase does not pose any problem in this case (megabit rates and TV-PCM).

The special case of $q = 4$ (quadriphase) of MPSK systems has been frequently used as a *multiplex system* to transmit two independent binary information flows simultaneously. Indeed, the original idea of Kineplex resulted from this requirement (Doelz et al 57). It has been revived for the transmission of two independent PCM channels (Anderson 63). The same idea leads also to the simultaneous transmission of two independent binary information streams via the two quadrature components of a subcarrier modulated in DSB (O'Neill and Saltzberg 66). If both channels are exactly synchronized and of equal amplitude and if exactly rectangular waveforms are used, the resultant signals are identical for DSB-MUX and for MPSK $q = 4$, the first group of encoding modes.

A critical circuit problem in MPSK receivers is the question of a limiter in the intermediate frequency (IF) amplifier versus a linear amplifier (Zabronsky 61). There is apparently no completely satisfying answer to this question. It depends on the channel, the type of information, the type of coding, etc. The limiter question has to be decided in the context of an integrated system.

Thus far we have discussed quaternary and octonary MPSK systems, yet theory showed that the highest utility

is for q = 3. Still we could not find a report in the literature about an actually developed ternary system, besides the *multi-lock system of the Robertshaw-Fulton Company* (Crafts 58), which used the third-phase state as a means to enforce sufficient spectral energy at the carrier frequency and not to carry information.

H4. *Separate Phase Reference MPSK Systems*

In the introduction to this section we divided the MPSK systems roughly into two large groups of systems : automatic phase reference (APR) systems, discussed in subsection H3, and separate phase reference (SPR) systems, to be discussed here. The SPR systems are the counterpart to the very successful MASK-SSB systems with pilot carrier(s) which we discussed in subsection F5. Yet in the MPSK case, until recently, much less attention has been given to systems with separate reference channel than to the previously discussed systems with automatic phase reference. Bussgang and Leiter (65a, 66) recently reviewed the analytical background of SPR-MPSK systems and referred to two different approaches for the transmission of the phase reference independently of the information.

The *first approach* uses one (or more) *reference tone* of constant frequency. Such special reference subchannels can be operated with a very small bandwidth, thus permitting the extraction of their centre frequency in the receiver with filters of phase-lock loops of very small noise bandwidth. The phase of this reference signal must be adjusted in the receiver to compensate for the frequency difference between the information carrier and the reference carrier and also for the different phase delay due to the different bandwidth in the two channels. The problem of finding the optimum power division between the information channel(s) and the reference channel(s) is critical (Stiffler 64b). A system which applies this first approach is the *DEFT system* developed by the *Electronics Division of General Dynamics Corporation* in Rochester, New York (DeHaas 64, 65). This is a multi-signal modem with many subchannels operating in frequency division in a voice frequency band of from 300 to 3000 Hz. Many reference frequencies are spread across the band, spaced 150 Hz apart. The information subcarriers are modulated in quaternary MPSK with an elementary length of 26 2/3 ms. The total modem is designed for 2400 or 3600 bps and it has been tested over a 7000-mile ionospheric path. Another system applying the reference tone approach is the *phase telemetry comparison system* used by the *U.S. Naval Space Surveillance System* (Easton and Downey 66). It uses 24 subchannels of 40 Hz bandwidth spaced 100 Hz apart. Only two reference tones are needed, since the system operates over telephone lines. The two-reference frequencies are placed at 1000 Hz and 1100 Hz. Their beat frequency is the 100-Hz timing and phase reference.

The *second approach* for the transmission of the phase reference uses phase multiplexing within a subchannel to transmit the phase reference. One quadrature component of the sine-subcarrier carries binary PSK signals, while the other phase quadrature component remains essentially unkeyed to serve as a phase reference. The *Kathryn modem* developed by *General Astrionics Corporation* (Bello 65a) shows an application of this approach. This modem modulates a large number of subcarriers in phase quadrature, whereby one component is modulated by the information in binary PSK while the other phase quadrature component is modulated by a binary pseudo noise (PN) sequence to serve as an accurate phase-and-timing reference. All subcarriers carry their own reference in this way. The quadricorrelator (Hannan and Olson 61) is an interesting variety of phase-lock loops which employs two phase detectors commonly known as the I (or in-phase) detector and the Q (or quadrature phase) detector. It may be useful in systems with quadrature reference channels.

A closely related idea was expressed by Renshaw (65a), who suggested the use (at specified time instants) of the phase difference between two subcarriers of different, but constant and well-related, frequencies to express the information and/or any phase reference.

The application of limiters in the receiver subchannels for either the information carrier or the reference tones is discussed by Zabronsky (61).

H5. *Applications of MPSK Systems*

Much of what has been said in the previous subsections already indicated that MPSK systems, like MFSK systems, find wide applications, primarily in time variable channels. The key problem is again the question of how to maintain accurate phase references over such time variable channels (Bussgang and Leiter 65b).

The statistical characteristics of fading channels was extensively explored in the past ; some of the important publications were reviewed in Part I, sections B2 and D2. Lutz et al (59), for example, experimentally explored prior to 1959, the phase distortions suffered by pulses of 1 ms and 20 μs length when passing ionosphere paths of 3000 km-length and beyond. They describe a data transmission system with phase change signalling that operated with millisecond pulses. Based on such knowledge it was possible for many research teams to lay out further statistical tests with the classical data transmission systems when operating under fading conditions and to attempt the design of more sophisticated systems with special antifading or antimultipath characteristics.

Glenn and Liebermann (62) experimentally analyzed the performance of DC-PSK in comparison with DSB quadrature operation and noncoherent FSK under fading and jamming conditions. Their results showed that " the DC-PSK system was most critical to fast fading, while the noncoherent FSK was somewhat better than the quadrature detection system ". They also recognized that " the shape of the spectral density of the signal fading can have a significant effect on error probability performance ". Fontaine (63) reported on comparisons between Kineplex and other data systems over telephone circuits carried via forward propagation tropospheric scatter circuits. The transmission of the modified diphase signals (section H1) over radio links was reported by Kolarck and Paramithas (63), and Ferrell and Matava (63) related their experience with a point-to-point DC-PSK system operating over MF radio (2 to 3 MHz).

The experience gained from the above-mentioned experiments and from many other observations leads to a number of analytical approaches and to some design efforts, all aiming at a solution of the crucial problem of getting digital data over a fading channel.

Lindsey (64) published a theoretical analysis of channels with " resolvable multipath and diversity reception ", for which he borrowed the term " multichannel " from an earlier paper of R. Price (62). Another theoretical analysis of Becker et al (64) was previously mentioned in this survey. The paper also contains an analysis of the transmission of DC-PSK signals over a fading tropospheric scatter circuit, which may be useful for further MPSK developments. Gerastovskiy (65) contributed an investigation of the noise immunity of phase telegraphy under Rayleigh fading conditions.

On the practical side, attempting the design of MPSK systems operating satisfactorily under fading conditions, we have previously referred to the DEFT design (Walker 64, 65, DeHaas 65, Greim et al 65) and to the Kathryn modem (Bello 65a, Bussgang and Leiter 66). From all these reports in the open literature this reviewer can reach only one conclusion : the problem of data transmission over fading (multipath) media is not yet solved. But much credit should be given to the research and development workers who have contributed to the understanding of this crucial problem. The result of all their efforts is, at least, a much better under-

standing of the complex natural effects that we are trying to master ; at most, the hope that we have already found the best design approach and that it is merely a matter of further small improvements to solve completely the problem of data transmission over fading channels.

It cannot be ascertained at this time if MPSK will be superior to other MXSK systems under all circumstances. The corresponding problem in analog transmission has never found a unique answer to the problem of which modulation method is the best. We still have AM, DSB-SC, SSB, VSB, FM, PhM and hybrid systems side by side. This reviewer believes the same will be the case with digital transmission modes. He further believes that binary systems in general, due to their simplicity, will continue to be used concurrently with nonbinary systems.

Part III—Coded Multidimensional Systems

Introduction

Part III deals with systems using a group (or a sequence) of digits as one unit, called a word. Each digit may assume one-out-of-q states of a well-defined waveform. Alternately during each digital interval the transmitter may select one-and-only one out of an alphabet of q different, but well-defined, waveforms. The systems discussed in Part III follow in the logical order of a multidimensional communications system. This order is explained in section I. According to the order, section J deals with codebooks, and with the receiving techniques. Section K discusses mathematical models of these monosignal multidimensional data transmission systems and section L reviews the experimental results that have been published so far.

Finally, Part IV is devoted to the waveforms that may be used in the implementation of monosignal nonbinary systems over bandlimited channels.

Some systems, which cannot be discussed in this paper, may be called polysignal systems because the restraint imposed in our present discussion, that in each time interval one-and-only-one waveform must be transmitted, is no longer in effect. These polysignal systems use the superposition (the simultaneous transmission) of many signals to achieve a higher information density. They form a rather complex " composite signal ". Many of the basic systems of Parts II and III are used in the polysignal systems on a subsystem basis in subchannels (frequency division) or in more general subspaces of the total signal space of the composite signal.

I. The General Multidimensional Transmission Mode

Part II of this paper presented the fundamentals and the state-of-the-art of one-dimensional nonbinary systems which we also called *multistate systems*. The definition of these systems indicated that a single parameter of a given waveform (ideally a rectangular pulse) was permitted to assume q different states. Thus the system offered a single degree of freedom, which is the reason for calling the systems described in Part II *one-dimensional systems*. It is evident, for example, that a single signal (pulse) cannot assume two different amplitude levels at the same time : their superposition would yield only a third and different amplitude level and the receiver would be in no position to recover the two original amplitude levels. This apparently is different when the two original amplitude levels are used to modulate two different, independent parameters of a more complex waveform. We speak in this case of a two-dimensional signal. Extending this idea to n-dimensions is the subject of Part III.

Whether one uses for the design of a multidimensional system sequences of a single type of waveform (Part III) or selections from an alphabet of band-limited waveforms (Part IV), one must apply rather complex receiving techniques to recover the information from the multidimensional signal. These techniques involve filtering or correlation devices, decoding circuits, and decision methods. These receiving techniques are discussed in the second part of section J.

The other sections of Part III, sections K and L, describe complete multidimensional systems. Their mathematical models are discussed in section K ; and their operational performance, in section L.

With monosignal operation in mind as the only restriction imposed on the multidimensional systems of Part III, we may now turn to fig. 16 to see the essential elements which a typical multidimensional system requires. We notice that a multidimensional system needs a codebook and a waveform library for its specification. The codebook (discussed in Part III) will be programmed into the transmitter encoder and into the receiver decoder. The waveform library (discussed in Part IV) will be programmed into the transmitter modulator matrix and into the receiver demodulator matrix.

Figure 16 shows the information flow beginning at the upper left side with a binary input. The serial-to-parallel converter takes a block of n bits into the encoder, where this input message is translated into the selection of one of the M* codebook words. Apparently, the minimum value for M must be 2^n, so that each of the possible input messages may get a code word assigned. Conversely, each code word, when identified by the decoder in the receiver, will uniquely correspond to one binary output message of n bits. It is, however, possible to make M larger than 2^n and to assign to some or to all the input messages not one but several code words. Some other code words may be service words not assigned to any input message. The reasons for such redundant designs may be the need for special synchronization words, privacy of transmission, addressing of special receivers, and so forth.

The codebook characters will preferably be selected from an alphabet of the same size as the waveform library. Evidently, the codebook may, in its simplest form, consist of a single row. This is a special case of a binary system with on-off keying of a rather complex but unique waveform determined by this single word of the codebook. The codebook characters may appear inside the equipment as multilevel pulses, as binary groups, or simply as the activation of different wires. Whatever the design may be, the effect must be that character A causes the selection of waveform A from the waveform library ; character B, the selection of waveform B ; and so on to character and waveform Q, which is assumed to be the last letter in the waveform alphabet (fig. 16).

Many multidimensional systems operate on the basis that the length of a codebook word corresponds to the number of timeslots (subintervals) that are provided for one transmission word (also called a transmission sequence). Again, exceptions are possible and synchronizing or addressing slots may be added in the modular matrix. The encoder will select one codebook word for each input word and forward it to the hold circuit to have it ready for translation into a sequence of transmission waveforms. This hold circuit

*In Part I we used " m " for the number of digits (states or levels) in a one-dimensional nonbinary system. We now use " M " for the number of words in a multidimensional codebook.

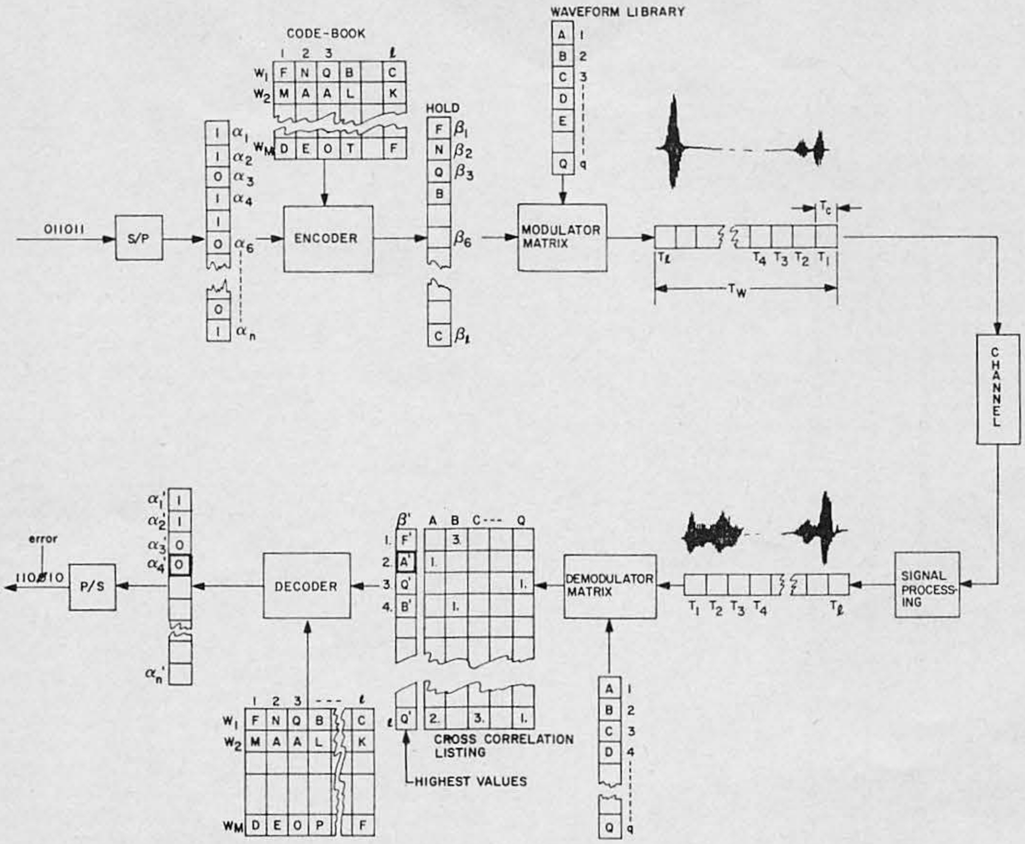


Figure 16.—Codebook and waveform library in multidimensional nonbinary systems.

may be necessary because either the encoder or the modulation matrix, or both, may need the complete message before they can begin their respective translations. The modulator matrix will select the correct waveform for each character held in the hold circuit and transmit the final sequence in time division into the channel. Several characters could be transmitted in frequency division (i.e., simultaneously) or with overlapping waveforms, but such systems fall under the concept of polysignal systems or multiplex systems and will not be discussed in the present paper.

Figure 16 shows that a codebook word will consist of l q-ary characters. The following relationship must hold to avoid any loss of information:

$$q^l \geqslant M \geqslant 2^n \tag{47}$$

If the equality sign is applicable on the left side of the equation, the codebook is complete. This means that all possible code words that can be formed with q-ary characters in l positions have been used as transmission code words. Such designs will be attempted when bandwidth is at a premium and the noise power in the channel is low. Systems of this kind will operate at a high bit density when selecting a highly bandlimited waveform library; i.e., they will operate on the right side of the utility chart (figs. 1 and 2 of Part I). On the other hand, a designer can made q and l so large that only a small number of all possible codebook words will be used. Such a redundant design will require more than the minimum bandwidth for the transmission channel. Indeed, many redundant nonbinary systems require more than the bandwidth of a nonredundant binary system, in which case they will operate on the left side of the utility chart. The advantage of the nonbinary mode of operation (as discussed in Part I) is higher utility.

In all cases where M (the number of codebook words) is much smaller than q^l, a major problem is the selection of the best subset of M codebook words from the complete set of all possible words. This is the same problem as the selection of error-correcting block codes. Much mathematical material is available in error-correction coding theory. Further details are given later in Part III.

The specification of a waveform library for the special case $l = 1$ (one-dimensional systems) has already been disdiscussed in Part II. There, it was assumed that all q waveforms differ only in one dimension; we called that case one-dimensional multistate systems. Special problems of waveform selection have also been discussed in Part I in subsections B1 and D1. From these reviews we may recall that the waveform library may be designed for high bit density, for medium bit density, or for low bit density. In the first case the product $B_c = BT_c$ will be made as small as possible (B is the essential bandwidth occupancy; T_c is the duration of a waveform). The smallest theoretical case is $B_c = 0.5$, the so-called Nyquist rate. The designer will try to make q larger than $2BT_c$, the maximum number of orthogonal waveforms. This is the rule in MASK and MPSK. In the second case (medium bit density) q will be close to β_c, the signal base of one character of the waveform library. Most designers will prefer to use orthogonal sets of waveforms. This is the case in MFSK and in special systems with closed orthogonal sets of waveforms. In the third case (low bit density), sets of waveforms with minimal cross-correlation coefficient will be preferred.

Following the information flow in fig. 16 through the channel, we first notice that the signals leaving the transmitter are grouped into words of length T_w, each consisting of l slots (or subintervals) of length T_c. Notice that in fig. 16 the words are always grouped like a train in the direction of the arrows; i.e., in the upper row the first time slot, T_1, is at the right side; in the lower row, at the left side.

On entering the receiver each transmission word is passed through the analog signal processor (Part I, C3 and D3). In the demodulator matrix each timeslot is synchronously correlated with all waveforms of the full library; the result of this operation is entered into the cross-correlation matrix where each row corresponds to one of the transmission slots and each column corresponds to one of the q waveforms. To illustrate this process we assume that the first codebook word has been transmitted. The first slot, T_1, contained the waveform character, F; the second slot, N, followed by Q and B until the lth character was transmitted as waveform C. In a noise-free and distortion-free test transmission, the cross-correlation matrix will have in each row only one very high positive value (autocorrelation coefficient, normalized to 1) and all other values will be much smaller (corresponding to the normalized cross-correlation coefficients). In the first row the 1 should be in the F column; in the second row, in the N column; and so on until the lth row, which should have a " 1 " in the C column. In an actual transmission, noise and distortions will modify this ideal situation. This is indicated in fig. 16 by postulating that the highest value in the first row would be in the F column (not shown); the highest value in the second row would appear in the A column (marked 1); and, in the lth row, in the Q column. Some systems will make, on the basis of the maximum value, a character-by-character decision right at this place or even inside the demodulator matrix. In this case they generate an error in the second row by deciding that A has been received instead of N, and another error in the last row. These errors may or may not be corrected later. More sophisticated systems will try to save the runner-up information, and carry on to the decoder at least the information about the second and third highest cross-correlation, leaving the final decision to the decoder. The most sophisticated systems will forward the complete cross-correlation matrix with all cross-correlation coefficients (Floyd and Nuttall 65). In each case the decoder will perform a second cross-correlation operation. In the simplest case it will take the tentatively decided word in column B′ (in our case, the word F′A′Q′B′ . . . Q′), correlate it with all M codebook words, and select that word which differs in the least number of positions. In a more complex, but also more efficient, receiver, the decoder will receiver the actual values of the cross-correlation coefficients of the winners in each row and use them for the calculation of the distance (section D1 in Part I) between the received noisy word and the closest clean words of the codebook that is stored in the decoder. In the most complex receiving system the whole cross-correlation matrix will be used by the decoder, which will reconstruct the noisy version of each possibly received word by using the actual cross-correlation coefficients of each character in each word. The decoder will then attempt to find the noisy word that has the smallest distance from any one of the M codebook words. It will decide that the word with the smallest such distance has been received.

The word the decoder has finally calculated as the most likely received word will be translated back into its binary equivalent with the help of the codebook. This is symbolized by the x' column at the left lower end of fig. 16. If the combined demodulator, decider/decoder process selected the wrong codebook word, there must be one or more errors in the binary output word when compared with the binary input word. The ratio of such errors to the total number of transmitted information elements (bits) is the final quality criterion of the system (section C1 in Part I). Apparently the number of final output errors will have something to do with the translation rules which are applied at the transmitter encoder when going from binary input words to codebook words. If neighbouring codebook words (words with small distance) correspond to neighbouring words, the total error probability can be minimized. This, and a number of other general problems of multi-dimensional systems, are discussed in the next three sections in the form of a review of the present research in this area.

J. The Encoding and Reception Process of Coded Multidimensional Systems

We discussed some of the fundamental encoding problems in section D1 in Part I. Here we intend to go deeper into the nonbinary coding and reception processes.

J1. *Binary to M-ary Translation Codes*

The translation in the encoder from binary to M-ary had the attention of many research workers for problems other than nonbinary information transmission. In connection with pulse code communications systems, F. Gray (53) received a U.S. patent on the now-famous Gray code in 1953. At about the same time, H. J. Gray, Jr., Levonian, and Rubinoff applied such codes to analog-to-digital converters (H. J. Gray, Jr. et al 53). A year later F. A. Foss (54) investigated the application of such codes to digital control systems. Flores (56) summarized the results of this particular branch of coding theory, with the help of elementary number theory, under the title: " Reflected Number Systems ". Since then reflected binary codes (as Gray codes are also called) have been the subject of more exhaustive research by Patterson (57), Gilbert (58), and Okunev (65, 64). The authors of these later publications clearly had in mind the application to nonbinary systems.

Gray codes, which belong to the unit distance number representation systems (Patterson 57), may be considered as cyclic codes or closed codes (Okunev 64). The term " unit distance " indicates that each code word differs only in one position from the two neighbouring code words when they are assigned fixed positions in a codebook. The example of a triple Gray code in fig. 17 makes this clear. The eight possible binary code words with three elements are arranged on the left side in the order W_1 to W_8. For example, the code word W_5 differs in only one position from the code words W_4 and W_6 next to it in the codebook. The right side of fig. 17 shows the cyclic or closed character of these codes which gives the first word W_1 and the last word W_8 the same advantage as all the other words. They are considered neighbouring words and as such differ in only one position.

These codes are of importance for nonbinary systems when selecting the translation mode from the binary input to the codebook of M q-ary message sequences of length l. The one-dimensional systems discussed in Part II especially demonstrate the interconnection between the class of waveforms and the encoding procedure. These systems operate, as explained, on a digit-by-digit basis. The codebook degenerates to a single column ($l = 1$). In this case it is particularly important that the most likely occurring demodulator errors cause the least number of residual binary errors in the output words. Gilbert (58) investigated codes with a total number of code words (M) which is not a power of " 2 ". For example, codebooks for $M = 10$ are very important. Notice, however, that reflected binary codes have no particular advantage in one-dimensional systems if there is no inherent unsymmetry in the q-ary transmission error probabilities of the various states. In certain PCM systems, for example, where the transmission signals are binary and where all errors are equally likely, Gray codes have no basic advantage over other codes (Filipowsky and Muehldorf 65a, Chapter 7, p. 575). Yet, they may be of importance in multidimensional nonbinary systems, even if all waveforms of the waveform library have equal error probabilities.

J2. *The Evolution of Block Codes (Codebooks)*

The *research on codebooks* has been active. Many mathematical models for suitable codebooks have been published since 1949 when Shannon postulated the conditions which designers must meet when striving for high efficiency systems. We discussed these general conditions in Part I (C2 and D1).

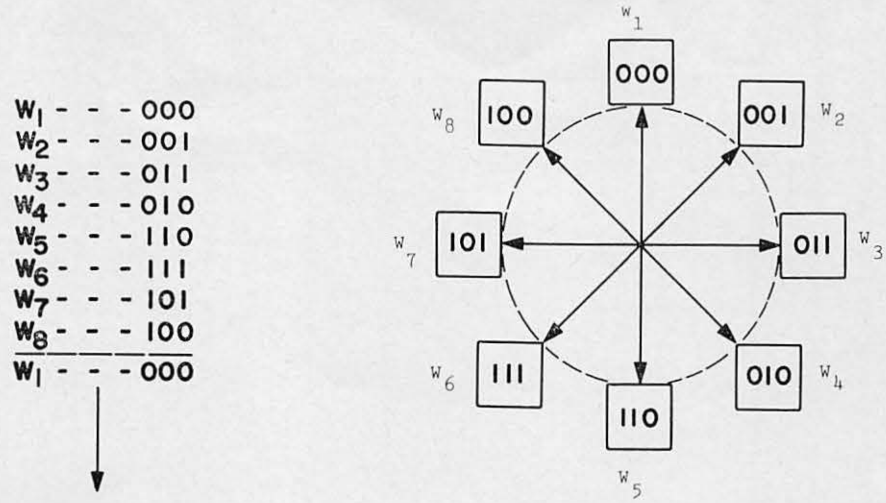


Figure 17.—*Examples of the triple Gray code.*

Here we mention the basic approaches to realizable block codes (codebooks) and, in the following sections, we discuss a number of actually worked-out codebooks.

The earliest attempt to specify a codebook in line with Shannon's recommendations is, most likely, the randomization approach published by Rice (50). He demonstrates that condition 3 of Shannon, the white noise character of the signals, can be met by a codebook where the letters for each code word are selected at random from a population of real numbers which has the same distribution as the noise. He investigated a Gaussian distribution and, in terms of our example in fig. 16, he would select the alphabet of q letters so that only a few would have very large positive or negative values and the majority would centre around zero, establishing a Gaussian amplitude distribution. From this population he would draw a letter (real number) for the first field (row 1, col. 1) of the codebook matrix, another real number for the second field (row 1, col. 2), and so on, until the first word is formed. The second, third, etc. word would be formed in the same way by picking values from a large Gaussian population at random until the codebook of M words is finished. Naturally, the receiver must get the complete codebook in advance of any transmission. Rice proved that a system on this basis can achieve Shannon's upper bound of the channel capacity, but only for M approaching infinity.

Gilbert (52) approached the same problem from the other end, building up codebooks of very small dimensions in such a way that all columns correspond to the coordinates of an l-dimensional Euclidian space. He specifies the best alphabets of two and three dimensions with codebooks from 3 to 13 words. Gilbert is plotting the performance of nonbinary systems of this kind in an " efficiency chart " very similar to the utility chart described in Part I. Indeed Gilbert's result can be easily plotted in a utility chart of multilevel systems (Filipowsky 68d).

Shannon (48) and (later) Gilbert (52) pointed out that a good alphabet can be obtained by using as code words the vectors to all the lattice points which are separated at least unit distance from one another and which are contained inside a hypersphere about the origin of the l-dimensional space. This sphere-packing problem elicited much interest in multidimensional geometry (Schaute 05 ; Coxeter 48, 62 ; Sommerville 58). Kharkevitch (56) and Poritsky (57) applied it in coding theory. Basore (59) and Stutt (59) developed on this basis the regular polyhedron codes, also called regular simplex codes. Stutt (60) explains a regular simplex in l dimensions as a general body bounded by $l + 1$ intersecting hyperplanes ; this is the analog of a triangle in a two-dimensional space or a tetrahedron in a three-dimensional space. If the edges are all of equal length, the simplex is regular.

Stutt (60) specifies the codebook in l-dimensional space as an M-member column matrix of code word vectors, which may also be expressed as a matrix product of an M-by-l matrix of projections and an l-member column matrix of unit coordinate vectors.

Balakrishnan (61) gave a rigorous mathematical analysis of the sphere-packing problem. Ziv (62, 63), based on Fano's (61) studies of the general discrete memoryless channel, investigated the coding and decoding schemes for the time-discrete nonbinary transmission system with codebooks of the random selection type, originally suggested by Rice (50).

A recent encoding process, called " linear-real coding ", published by W. H. Pierce (66) also uses a codebook matrix with real numbers. In this case, the input information has been systematically smeared out over the whole transmission interval. The transmission signals do not represent one-and-only-one code word during a given transmission interval, but actually a superposition of all code words. In our terminology we have to classify the linear-real coding process as a polysignal system and not as a single-signal system.

Contrary to Pierce's extremely general approach one can find in another recent publication by Slepian (65a) a rather simple but still very efficient encoding procedure. It is called *permutation modulation* because the codebook is defined by a single codebook word, placed as W_1, and by all possible permutations of this first word, placed as W_2, W_3, . . . , W_M. Figure 18 shows a number of typical examples of permutation codebooks. On top of the figure is the general form of a word. The μ_i elements can be any real numbers, not necessarily integers. Zero and negative values can be included. Each number can be repeated any number of times. There are two types of these codes :

(i) Type I is formed by placing the first code word into its natural order, i.e., lowest numbers to the left and progressing to the right with increasing numbers, equal numbers being next to each other. All the other code words are formed by all possible permutations of the first code word. Code numbers 1, 2, 3, belong to this type.

(ii) Type II codes start in the same way with the exception that the first word consists of non-negative real numbers only. The other codes are then formed by permuting the order of the elements and by making all possible assignments of signs to the elements. Code numbers 4 and 5 are of this type.

$$W_1 \ldots \{\overbrace{\mu_1\ \mu_1 \ldots \mu_1}^{m_1}\ \overbrace{\mu_2 \ldots \mu_2}^{m_2} \ldots\ldots \overbrace{\mu_k \ldots \mu_k}^{m_k}\}$$

$\mu_1\ \mu_2 \ldots \mu_k$ real numbers;
$n = m_1 + m_2 + \ldots + m_k$

Code #1

$n = 3;\ k = 3$

W_1 ...	1	2	3
W_2 ...	1	3	2
W_3 ...	2	1	3
W_4 ...	2	3	1
W_5 ...	3	1	2
W_6 ...	3	2	1

$m_1 = m_2 = m_3 = 1$
$M_I = 3! = 6$

$$\boxed{M_I = \frac{n!}{m_1!m_2!..m_k!}}$$

Code #2

$n = 4;\ k = 3$

W_1	-1	-1	+1	+2
W_2	-1	-1	+2	+1
W_3	-1	+2	-1	+1
W_4	-1	+2	+1	-1
W_5	+2	-1	-1	+1
W_6	+2	-1	+1	-1
W_7	+2	+1	-1	-1
W_8	+1	-1	-1	+2
W_9	+1	-1	+2	-1
W_{10}	+1	+2	-1	-1
W_{11}	-1	+1	-1	+2
W_{12}	-1	+1	+2	-1

$m_1 = 2;\ m_2 = m_3 = 1$

$M_I = \frac{4!}{2!} = 12$

Code #3

$n = 4;\ k = 2$

W_1	-1	-1	-1	3
W_2	-1	-1	3	-1
W_3	-1	3	-1	-1
W_4	3	-1	-1	-1

$m_1 = 3;\ m_2 = 1$

Condition for Simplex:
$m_1\mu_1 + m_2\mu_2 = 0$

$M_I = \frac{4!}{3!} = 4$

Normalized code

$\mu_1 = -\frac{1}{\sqrt{12}};\ \mu_2 = \frac{3}{\sqrt{12}}$

Code #4

$n = 4;\ k = 2$

W_1	0	0	0	1
W_2	0	0	1	0
W_3	0	1	0	0
W_4	1	0	0	0
W_5	0	0	0	-1
W_6	0	0	-1	0
W_7	0	-1	0	0
W_8	-1	0	0	0

$m_1 = 3;\ m_2 = 1$
$h = n - m_1 = 1$

$$\boxed{M_{II} = \frac{n!2^h}{m_1!m_2!\ldots m_k!}}$$

$h = n - m_1$ if $\mu_1 = 0$
$h = n$ if $\mu_1 \neq 0$

Code #5

$n = 3;\ k = 1$

W_1	1	1	1
W_2	1	1	-1
W_3	1	-1	1
W_4	1	-1	-1
W_5	-1	1	1
W_6	-1	1	-1
W_7	-1	-1	1
W_8	-1	-1	-1

$m_1 = 3$
$h = n = 3$

$M_{II} = \frac{3!.2^3}{3!} = 8$

Type I: $\mu_1 < \mu_2 < \mu_3 \ldots < \mu_k$ (Codes #1–#3)

Type II: $0 \leq \mu_1 < \mu_2 < \mu_3 \ldots < \mu_k$ (Codes #4–#5)

Figure 18.—Examples of codebooks for permutation modulation.

The two equations for M_I and M_{II} give the size (total number of code words) for each codebook.

It is interesting to note that many well-known codes are special cases of such permutation codes. Code number 3, for example, is a simplex code with one redundant dimension. Slepian points out that the condition,

$$M_1\,\mu_1 + M_2\,\mu_2 = 0 \quad , \tag{48}$$

makes a code with $k = 2$ (two different characters) a simplex code.

Code number 4 is a biorthogonal code, since the cross-correlation between any two words is zero, except for the pair formed by a code word and its negative (W_1 and W_5, for example). Such pairs have a cross-correlation coefficient of -1. If a code is formed from the same first word by permutation only (Type I), a codebook of the first four words results. This is an orthogonal code and leads particularly to a nonbinary system that we called multiple time shift keying, also known as quantized pulse position modulation. That and other kinds of orthogonal and biorthogonal codes will be discussed in section K.

Code number 5 is the well-known binary PCM code with three digits. It results here as a subclass of Type II permutation codes. Slepian made a search of about 100 codes, some of very large n (up to 100), and investigated their performance in ideal low-pass systems, comparing such systems with the upper bound of Shannon's channel capacity.

A report by Floyd and Nuttall (65) contains a very general treatment of nonbinary systems from which we represented a specialized version in fig. 16. This specialized version is called by the authors the q-M-ary system, indicating that they use a codebook of M words consisting of sequences of q-ary characters. The largest part of the Floyd and Nuttall report deals then with a further specialization, making $q = 2$. This means they investigate nonbinary systems using waveforms which are synthesized from binary sequences (code words). We would like to stress the importance of their report to research workers investigating more general codebooks, similar to those discussed before, but using integers as codebook characters.

The special selection of signals for the waveform library will be discussed separately in Part IV because it seems that two different schools of thought are simultaneously developing rather different and novel ideas. Here in Part III we intend to concentrate on the codebook approach only, i.e., on the effort to express messages in a sequence of numbers without regard to the ways of expressing these numbers in band-limited waveforms.

J3. *Receiver Fundamentals for Multidimensional Systems*

Turning now to the *receiver side* of a multidimensional nonbinary system, one notices the large amount of research that has been devoted to the specifications of an optimum receiver design for analog, binary, and nonbinary signals. We already explained in Part I that an efficient receiver for nonbinary transmission signals will have at the input a rather complex analog section, called the signal processor or signal extractor. It is in this analog section that much of the research results of optimum reception theory for analog signals can be applied—with reservations, however. We mentioned a selection of references to research on mathematical extractor models in Part I, subsection D3. Part of the devices used in those mathematical models will be located in the block called *demodulator matrix* in fig. 16. This demodulator matrix will assume quite different forms for the different nonbinary systems and for the different waveform libraries in use. Some of its realizations have already been discussed in Part II in connection with the various multistate systems. Regardless of the specific circuits in a multidimensional system, the output of the demodulator subsystem must give the best possible estimate of which one of the waveforms has been received in any subinterval T_i out of the l intervals of a transmission word.

To perform an efficient estimation operation, the demodulator subsystem must receive the following information as input:

(i) The complete waveform library, with clean undistorted noise-free waveforms.

(ii) The noisy mutilated waveforms, as received over the channel, and after having been processed in the signal processor.

(iii) Timing information indicating with highest precision the beginning and the end of each subinterval and any additional phasing information of carriers and subcarriers.

(iv) Any additional statistical information (long term and short term) about the status of the channel (noise, fading, etc.), hopefully tagged with confidence levels.

(v) In advanced systems, any additional service information, such as code numbers, indications of message repetitions, etc.

Some of these inputs will be permanent or semipermanent and thus may be stored in memories or in the form of circuits (matched filters) within the demodulator subsystem. The waveform library falls into this group. Details will be discussed in Part IV. Other inputs will be transmitted over the channel as special service messages ; they will become available to the demodulator subsystem via the decoder (service information). Still other inputs will be available from special circuits in the receiver, but the operation of these circuits must be maintained by continuously or stepwise updating their status (synchronization via phase lock loops).

Finally, it will be necessary to apply special sensors and computing circuits to extract the statistical information about the channel either directly from the noisy channel, or indirectly via demodulator and decoder from the noisy waveforms and from corrected message errors.

The basic nonbinary systems discussed in Part II do not yet provide all this information in satisfactory form. Yet the mathematical models used in the analysis of these basic systems are derived under the assumption that all this information will be provided. This discrepancy between theory and practice presents several problems to the practical engineer. To bridge this gap, further research about the receiver subsystems is needed.

No matter what degree of sophistication the demodulator system assumes and how close it is able to approach an ideal operation, the output of the demodulator system will always be a complete or a simplified cross-correlation matrix as shown in fig. 16. If the receiver can be linked to the transmitter via an ideal attenuator (noise-free case), the cross-correlation matrix should be an ideal matrix containing " sure " coefficients. This ideal matrix, in the case of other codes than simplex codes, will not be as regular as the one for simplex codes (Floyd and Nuttall 65).

The noise-free, distortion-free case is not a realistic model for practical systems. It is evident that the designer must start off with some reasonable assumption about the noise level at the output of the demodulator subsystem. If the characteristics of the waveforms (ratio b_D/q) and the signal contrast (σ) at the receiver input, together with the noise statistics, permit the prediction of a high signal contrast at the demodulator output, one will be inclined to permit a positive high value of the maximum cross-correlation coefficients, counting on small variations of the output pattern due to the noise. In other words, if the external noise is small, one can permit a certain amount of internal cross talk noise between the waveforms. Conversely, if the signal contrast which the designer expects at the output of the demodulator system is small (i.e., if the noise is strong), one will attempt to make the maximum cross-correlation coefficients in the matrix as small as possible (e.g., zero or slightly negative). This will leave a wider margin for the noise before final errors in the decision system can result.

Turning now to the *research on the decider and decoder subsystem*, one will clearly recognize that, the simplest transfer mode of extractor results after a maximum likelihood decision needed an accurate analysis. In this case a multivariate decision is made at the end of each interval T_D. Nuttall (62) derived an expression for the error probability of such a decision, based on the maximum likelihood decision rule. He assumes that the demodulator matrix in fig. 16 performs, during each waveform interval T_D, a cross-correlation of the received noisy signal with replicas of all possible waveforms. Assuming that the first signal has been transmitted, the demodulator will receive $r(t) = s_1(t) + n(t)$, with $n(t)$ representing the noise in T_D. The demodulator matrix will have q outputs of the form :

$$\left.\begin{aligned} &\text{Output one :}\\ &x_1 = \int_0^{T_D} s_1(t)\, s_1(t)dt + \int_0^{T_D} s_1(t)\,.\,n(t)dt \\ &\text{Other outputs}\\ &x_i = \int_0^{T_D} s_i(t)\, s_1(t)dt + \int_0^{T_D} s_1(t)\,.\,n(t)dt \end{aligned}\right\} \quad (49)$$

Assuming that all waveforms are of equal energy, E_D, and equally correlated, one can substitute :

$$\left.\begin{aligned} x_1 &= E_D + y_1 \qquad & y_1 &= \int_0^{T_D} s_1(t)\,.\,n(t)dt \\ x_i &= E_D\,\lambda + y_i \qquad & y_i &= \int_0^{T_D} s_i(t)\,.\,n(t)dt \end{aligned}\right\} \quad (50)$$

The decision rule implies that a correct decision will be made as long as $x_1(t)$ is larger than any other $x_i(t)$. The probability for a correct decision is therefore :

$$P_c = \Pr(E_D(1-\lambda) + y_1 > y_2, y_3, \ldots, y_q) \quad (51)$$

P_c can be calculated by integration over the joint probability density function,

$$P_c = \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{E(1-\lambda)\ y_1} \cdots \int dy_2 \ldots dy_q \,.\, p(y_1, y_2, \ldots, y_q) \quad (52)$$

Under certain assumptions it is possible now to calculate the probability of a correct decision under the MLD rule (Nuttall 62). The result is equation 53 :

$$P_c = \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} e^{-z^2/2}\,.\,P^{q-1}\left(z + \sqrt{\frac{E_D(1-\lambda)}{2N_o}}\right)dz$$

with (53)

$$P(x) = \frac{1}{\sqrt{2}} \int_{-\infty}^{x} e^{-t^2/2}\,dt$$

The function $P(x)$ is tabulated in many mathematical tables as the cumulative distribution function of the normal distribution. The integral of equation 53 is tabulated in the form :

$$\int_{-\infty}^{+\infty} \phi(z)\, P^{q-1}(z + a)\, dz \quad (54)$$

in the following references (Birdsall and Peterson 54, Urbano 55, D. M. Green and Birdsall 58, Elliott 59).

The average error probability for one character to be in error is :

$$\bar{e}_c = 1 - P_c \quad (55)$$

Another approach to the same problem has been published by Arthurs and Dym (62) in the same year as Nuttall's paper. These authors arrive at an upper and lower bound for the character error ratio, $\bar{e}_c$. They use the distance between message points as the key variable, while Nuttall uses the cross-correlation coefficient λ as the key variable which expresses the characteristics of the waveform library. A direct comparison between the results of the two indepen-

dently developed theories requires, therefore, the knowledge of the actually used waveforms $s_1(t) \ldots s_q(t)$. More research will be necessary to perform such comparisons. When they become available it will be useful to plot the results in a utility chart and to compare them for any given dimensions, $l = 2BT_D$, with the ideal time-limited system of Shannon/Slepian which has the same number of dimensions. (The utility curves for this ideal time-limited system are in fig. 5 of Part I.) Such comparison would disclose the efficiency of a nonbinary system with the particular waveform library.

There are less definitive results in the literature regarding an accurate analysis of the other, more efficient, modes of transfer of the demodulator results to the decoder.

The *decoder research* followed, for a long period, the development of binary decoders. Many interesting nonbinary error-correcting codes are known but decoding algorithms have not been devised, except in very simple cases. Floyd and Nuttall (65) present a general approach to the decoding problem in their treatment of the q-M-ary nonbinary system. This follows closely the lines of fig. 16 ; full use is made of Nuttall's analysis (62) of the MLD demodulator which we discussed above. Their final decoder is again a maximum likelihood decoder. Floyd and Nuttall developed an upper bound for the final error probability at the output of the decoder.

An interesting recent development in the decoding technique is the procedure for decoding Slepian's permutation codes. We explained these codes in connection with fig. 18. The particular regular structure of a codebook derived by permutations of the first code word leads to a very simple decoding algorithm (Slepian 65a).

The decoder for Type I codes converts a received sequence $\beta'_1, \beta'_2, \ldots, \beta'_l$ (fig. 16) into a word of the codebook by replacing the i-th largest β' by the i-th largest number in the sequence forming the first word of the codebook. The decoder for Type II codes converts a received sequence $\beta'_1, \beta'_2, \ldots, \beta'_l$ into a word of the codebook by replacing the β' that is i-th largest in absolute value by the i-th largest number in the sequence forming the first word of the codebook. The j-th term in the sequence thus formed is then given the algebraic sign of β'_j ($j = 1, 2, \ldots, l$).

In fig. 19 we show an example fo this decoding operation, comparing it with a maximum likelihood decoding operation in the simplest mode (decision after demodulator) and in a more complex mode (largest real value taken from the demodulator for each β'_j). Figure 19 shows at the top the transmitted code word, which is W_8 taken from code number 2 in fig. 18. It is assumed that the code word arrives in heavily mutilated form as W_x at the output of the demodulator matrix. If a decision is made after the demodulator, the code word W_y would result, provided the thresholds are at 0 and 1.5, halfway between the three acceptable digits -1, $+1$, and $+2$. If this were the end of the receiving operation, one can see that only two digits out of four would be correct, though an error-detecting device could be actuated, indicating that the resulting word is not a word of the codebook.

Figure 19 further shows, on the left side, the 12 words of the selected codebook. This is a rather arbitrarily selected codebook and the reader can verify that its noise-free cross-correlation matrix shows cross-correlation coefficients from $\lambda_{4,8} = -1$ to $\lambda_{5,8} = +6/7$. Such large differences may be desirable in special cases, especially when the input message probabilities also show very large differences. The second column shows the decoding matrix when decoding the word W_y in an MLD operation. The products of corresponding digits are listed. For example, the cross-correlation of word W_y with the word W_1 is performed by multiplying the first digit of word W_y (-1) with the first digit of W_1 (-1). The result, $+1$, is noted down. The second product is $-1 \times -1 = +1$; the third product, $-1 \times +1 = -1$; the fourth product, $+1 \times +2 = +2$. The sum of the four products $(+3)$ must be divided by the energy of a clean signal This is the sum of the squares of all four digits of any codebook word, in this case, $E = 7$. The cross-correlation coefficient λ_{y1} is, therefore, $3/7 = 0.429$.

The cross-correlation of W_y with the whole codebook shows that there are three coefficients of equal and highest positive value. The third matrix in fig. 19 shows the same procedure when applied to the real value (undecided) received word, W_x. In this case the correct word can be clearly identified because λ_{x8} has the highest positive value of all cross-correlation coefficients $(3.6/7 = 0.514)$. At the right side of fig. 19 Slepian's decoding procedure is applied ; it too leads to the correct result. However, this simple algorithm can be much more easily instrumented than the MLD decoding operation, particularly when the codebook

Transmitted word . . . . . . W_8	+1	-1	-1	+2
Received at output of decoder W_x	-0.2	-2.5	-0.3	+0.5
Decision after demodulator . W_y	-1	-1	-1	+1

Threshold settings:

0	+1.5

Codebook

W_1	-1	-1	+1	+2
W_2	-1	-1	+2	+1
W_3	-1	+2	-1	+1
W_4	-1	+2	+1	-1
W_5	+2	-1	-1	+1
W_6	+2	-1	+1	-1
W_7	+2	+1	-1	-1
W_8	+1	-1	-1	+2
W_9	+1	-1	+2	-1
W_{10}	+1	+2	-1	-1
W_{11}	-1	+1	-1	+2
W_{12}	-1	+1	+2	-1

ML Decoding of W_y

W_y	-1	-1	-1	+1	λ_{yi}	Rank
λ_{y1}	+1	+1	-1	+2	+3/7	1.
λ_{y2}	+1	+1	-2	+1	+1/7	
λ_{y3}	+1	-2	+1	+1	+1/7	
λ_{y4}	+1	-2	-1	-1	-3/7	
λ_{y5}	-2	+1	+1	+1	+1/7	
λ_{y6}	-2	+1	-1	-1	-3/7	
λ_{y7}	-2	-1	+1	-1	-3/7	
λ_{y8}	-1	+1	+1	+2	+3/7	1.
λ_{y9}	-1	+1	-2	-1	-3/7	
λ_{y10}	-1	-2	+1	-1	-3/7	
λ_{y11}	+1	-1	+1	+2	+3/7	1.
λ_{y12}	+1	-1	-2	-1	-3/7	

ML Decoding of W_x

W_x	-0.2	-2.5	-0.3	+0.5	λ_{xi}	Rank
λ_{x1}	0.2	2.5	-0.3	1.0	3.4/7	2.
λ_{x2}	0.2	2.5	-0.6	0.5	2.6/7	
λ_{x3}	0.2	-5.0	0.3	0.5	-4.0/7	
λ_{x4}	0.2	-5.0	-0.3	-0.5	-5.6/7	
λ_{x5}	-0.4	2.5	0.3	0.5	2.9/7	3.
λ_{x6}	-0.4	2.5	-0.3	-0.5	1.4/7	
λ_{x7}	-0.4	-2.5	0.3	-0.5	-3.1/7	
λ_{x8}	-0.2	2.5	0.3	1.0	3.6/7	1.
λ_{x9}	-0.2	2.5	-0.6	-0.5	1.2/7	
λ_{x10}	-0.2	-5.0	0.3	-0.5	-5.4/7	
λ_{x11}	0.2	-2.5	0.3	1.0	-1.0/7	
λ_{x12}	0.2	-2.5	-0.6	-0.5	-3.4/7	

Slepian decoding

1. Replace 0.5 by 2.0
2. " -0.2 by 1.0
3. " -0.3 by -1.0
4. " -2.5 by -1.0

Decide:

+1.0	-1.0	-1.0	+2.0

Find from codebook:

$\boxed{W_8}$

Figure 19.—Comparison of decoding procedures for permutation modulation.

is very large. In his 1965 paper Slepian (65a) develops explicit expressions for the error probabilities of Type I and Type II codes and submits proof that the decoding procedure described above and demonstrated in fig. 19 minimizes the average error probability per transmitted word under conditions specified in the paper.

Concluding section J we should refer to an applications-oriented report (Brumbaugh 60) by staff members of Radio Corporation of America. Their study on coding and modulation techniques for future Astro-Communications systems compares open-loop redundant codes, open-loop book codes, and closed-loop coding techniques. It gives information about single level and multilevel phase shift keying and frequency shift keying methods of modulation. The study concludes that the most promising coding techniques are book codes with orthogonal signals combined with multilevel FSK and differentially coherent PSK waveforms. The more advanced nonbinary systems had not yet been considered in this study of August 1960.

K. Mathematical Models of Multidimensional Systems

The present section K shows how mathematical models of these coded multidimensional systems can predict their efficiency. Although not many such systems are in an operational status, a few will ultimately emerge as the highly efficient systems of the future. The scarce experimental results which are now available will be discussed in section L.

K1. *The Evolution of Mathematical Models for Coded Systems Prior to* 1957

The previously mentioned dual approach to develop mathematical models of multidimensional systems, either along the coding idea or along the waveform library idea, is very much a historical fact.

The *coding approach* has its origin in the error-detecting codes for teleprinters that emerged in the second quarter of this century. In particular, the evolution of the seven-unit code (Sparks and Kreer 47), with its constant 4 : 3 ratio between spaces and marks in the teleprinter characters, paved the way towards a systematic investigation of binary codebooks. Hamming (50) of Bell Telephone Laboratories worked for many years on an extension of the error-detecting parity check codes—previously used in the early computing machines (Alt 48)—to error-correcting codes. Shannon, in his famous 1948 publication, credited R. W. Hamming with what thus far is the first error-correcting code, a block code of seven binary digits, including three check digits. Golay (49a) published a short note on digital coding, showing that Hamming's seven-bit code can be extended to blocks of 2^n-1 length, thus providing codes of 15, 31, 63, etc. binary elements. Still more important for the purpose of this paper is Golay's extension of Hamming's block codes to blocks of nonbinary elements. In the same year also appeared a mathematical paper by Foth (49) that could very well be the oldest contribution to the famous sphere-packing problem which in turn leads to the simplex codes (section J, especially Balakrishman (61)). Finally, in 1950 Hamming's famous paper on " error-detecting and error-correcting " codes defined many magnitudes which are characteristic for all block codes and which are essential for the application of binary sequences in nonbinary systems.

Whereas Hamming's paper concentrated on bit-by-bit decision and on logical operations for correcting errors, R. M. Fano in a report in 1951 and in a paper in 1952 prepared the way for the correlation detection of complete code words in place of the bit-by-bit decision followed by logical decoding. This step clearly converts binary data transmission systems with block codes to nonbinary data transmission systems by considering each transmitted code word as one transmission signal out of a collection of more than two transmission signals. The fact that the transmission signal consists of binary elements is merely a design feature and not a fundamental characteristic of the system.

These first results of the coding researchers led to the hope that the way to reach Shannon's channel capacity C (section C2) could be via such sequences of binary elements. We already referred in sections A and D1 to the research results of Shannon (48) and Rice (50), proving that the ideal channel capacity C (equation 17) can be reached in the limit when using, as transmission waveforms, sequences of increasing length consisting of real numbers drawn from a normal universe. The problem facing the mathematicians in 1952 was to devise an algorithm for artificially creating such random signals, i.e., to generate pseudorandom signals. In 1954 Golay announced his results on binary coding, indicating that *symbol-correcting codes* such as the Hamming codes are less likely to reach C than *message-correcting codes*. The former class of codes uses special check digits in addition to information digits ; i.e., they are defined by sets of logical equations for calculating the check digits. The message-correcting codes, on the contrary, are defined by a codebook ; there is no need to calculate any special redundant digits. Also in 1954 Elias showed that the symbol-correcting codes (also called systematic codes or digit codes) can be extended by an iteration of simple error-correcting and detecting codes. This process is easy to instrument and, by increasing the number of check digits and the complexity of the process, the designer can reach—for any given noisy channel—an error probability as low as he may desire. Elias called the process " error-free coding ". Yet a system using these codes cannot reach this error-free operation at channel capacity C. Feinstein (54, 55) and Shannon (56) proved that this *zero-error capacity* is not only a characteristic of Elias's check digit codes but a very fundamental magnitude that can be computed for many channel models. Reed (54) showed that a special class of message-correcting codes has multiple error-correcting capabilities. These codes are now well known under the designation of Reed-Müller (RM) codes, Reed having based his algorithm on the somewhat earlier work of Müller (53). Reed should also receive credit for introducing a *majority testing decoding* procedure, now better known as maximum likelihood decoding (section M) and closely related to the multiple-alternative detection method of Middleton and Van Meter (55).

Studies during these first ten years (1947-1957) of the evolution of the coding theory concluded that :

(i) Encoded sequences of numbers are useful as transmission messages.

(ii) The decoding process should be a maximum likelihood decoding (MLD) process rather than a bit-by-bit decision process with subsequent logical error correction.

(iii) The length of the code words (number of columns in the codebook matrix) should be as long as economically feasible.

(iv) The size of the code word alphabet (number of rows in the codebook matrix) should be matched to the SNR of the channel.

(v) When the right code and the optimum dimensions of the codebook matrix (ratio of the number of columns to rows) are selected, it is possible to arrive at error-free operation.

The unsolved problems, however, were numerous :

(i) Are binary sequences or nonbinary sequences preferable ?

(ii) What algorithms are available to design codebooks ?

(iii) How close can the transmission rate approach the channel capacity for any class of systems (special selection of codes, basis of the number system, selection of the decoding and decision procedure) ?

(iv) How far will the performance of any given system be below the ideal system when a maximum error ratio is specified ?

(v) What is the cost (circuit complexity) and the reliability of any design coming close to the optimum performance ?

When summarizing the efforts prior to 1957 to find suitable waveform libraries for multidimensional nonbinary data transmission systems (section M), one must conclude that no single approach was available that could directly lead to an engineering model. Yet there were many serious efforts to combine the coding approach with the waveform library approach to create the concept of a truly efficient system. These overall efforts are represented by two books and by two review reports.

The book by R. M. Fano (61) reflects in consolidated form the progress of both approaches during the 1950's. During this decade Professor Fano made significant contributions toward multidimensional data systems, both as professor and as research worker at Massachusetts Institute of Technology in Boston. The second book is the theory of optimum noise immunity by the Russian engineer and scientist, V. A. Kotel'nikov (59a).

The two review reports are the report by Nuttall (62) with 72 references and a more recent report by Gibson (66), with many charts suitable for practical tradeoff calculations.

All the efforts discussed above laid the ground work for a number of more definitive mathematical models of systems based on large codebooks.

K2. *Mathematical Models Since* 1957

The most important of the five unsolved problems that we listed as being attacked around 1957 was the search for algorithms to encode the input information into suitable codebooks. This search concentrated on binary sequences, since their behaviour was best known and since the binary logical circuit technique was well developed. In terms of fig. 16 this means that only the letters A and B would be used as elements of the codebook and of the decoder matrix. The waveform library would have only two members (A and B); the cross-correlation matrix in the receiver would have only two columns, marked A and B. We recall that one of the problems that had been solved prior to 1955 was the question of the optimum decoding procedure. Theory had proved that MLD or decoding by correlation was the optimum decoding mode.

To implement a data transmission system that uses binary code words as signals, the designer makes a number of selections. Most important is the selection of the code length (l), i.e., the number of binary elements (also called chips) in each code word. Also important is the selection of the number of code words (M). The ratio of l/M indicates the redundancy of the code; it is a measure for the number of "forbidden" words. If $M = 2^l$, the codebook is complete: all code words that can be formed from l chips are in the codebook. Naturally, such a system would be identical with an uncoded binary system. If $M = l$, the codebook matrix is a square matrix. It will be shown below that very useful orthogonal codes are known in this form. Most of the known systematic codes (error-correcting codes) have $l < M < 2^l$. They can be used in the manner indicated above and will operate as message-correcting codes.

The key problem to be solved before any design can start is naturally the *exploration of all possible codebooks and their encoding algorithms*. Since 1955 many research results have been published. The most general results, or those of special tutorial value, will be mentioned in this section.

By 1955 it was well known that periodical binary sequences could be generated with the help of shift registers and a binary feedback logic. Zierler (55) summarized the theory of such shift register generators and Golomb (55) investigated the randomness properties of the sequences that could be generated with the devices. Some sections of this classical report deal also with sequences of nonbinary chips. Huffman (56) established a theory for the encoding of a finite sequence of binary information digits with the help of a linear binary sequence filter. At the transmitter the information sequence is inserted into the filter and is immediately followed by another sequence; for example, an all-zero sequence, which is known to the receiver in advance of the transmission. The output of the filter becomes the sequence to be transmitted. An inverse filter is utilized at the receiver and the deviation of the additional sequence from its (*a priori*) *standard* form can be utilized for the subsequent correction of transmission errors. Thus Huffman succeeded in relating the codebook to the structure of a digital filter and to the use of an additional binary sequence known to both stations. Huffman also discusses an extension of this idea to other than modulo-two number systems. Sequences with nonbinary chips may therefore use the same basic procedure, provided that nonbinary logical elements are available. Birdsall and Ristenbatt (58) prepared an excellent tutorial introduction to linear shift-register-generated sequences.

In 1959 we find strongly increased interest in shift register sequences and in their theory (Zierler 59). Certain properties make them attractive for communication systems (Campbell 59), and renewed progress in the theory of linear sequential networks (Elspas 59) prepares the way to close the gap between the theory of systematic codes (Abramson 59) and the theory of block codes and group codes (Fontaine and Peterson 59, Slepian 60). The latter class is primarily destined to serve as codebooks in multidimensional systems. The first book on error-correcting codes was published by W. W. Peterson (61). Although basically concerned with the theory and the procedures for bit-by-bit binary symbol decoding, it is valuable for multidimensional nonbinary systems as a review of the coding theory of the 1950's. Independently but concurrently published, we note the first part of a comprehensive investigation of cyclic codes by Elspas and Short (61), performed under government contract. The final report was finished in 1962. The first comprehensive review of the results of the codebook approach as a means to create a large alphabet of waveforms in the form of binary sequences was also published as a report by the Jet Propulsion Laboratory in Pasadena, California (Baumert et al 61). An expanded version of this report was later made available in book form (Golomb et al 61). These surveys of the state-of-the-art in the theory of error-correcting codes and of the evolution of the binary codebook approach toward multidimensional communication systems paved the way for specialized investigations of some crucial problems. Eisenstadt (61) concentrated on the autocorrelation characteristics of pseudonoise sequences; Wolf (63) explored the effects of filtering a random binary sequence with certain finite memory filters (both linear and non-linear); Roberts (63) consolidated the information about linear and nonlinear shift-register generators; and Gilbert (63) considered the application of binary sequences as carriers in asynchronous multiplexing systems, arriving at a class of cyclically permutable error-correcting codes for this purpose.

Since 1963 many excellent papers have been available; some of them are applications oriented, while others are still exploring the remaining theoretical problems. A few contributions of general interest are mentioned here.

McWilliams (64) treats the practical codebook design problem of locating cyclic codes with a fixed block length and a fixed number of parity checks, provided such codes exist. A handbook of communication codes is published (64a), the synthesis of binary sequences with arbitrary realizable autocorrelation functions is attacked (Levitt and Wolf 65), a systematic classification of sequence-generating methods into five basic classes emerges (Brothman et al 65), and cyclic error-locating codes are compared with

many other known codebooks for their merits in multidimensional communications systems (Goethals 67).

We recently experienced the beginning of an investigation of orthogonal codes with nonbinary elements (S. H. Chang 66a). More will be said about this important branch of multidimensional nonbinary systems in subsection K7.

The references in section K which were discussed in historical order give a fair cross section of the vast literature on the research in codebooks and coding (or decoding) algorithms. This research resulted in specific mathematical models, discussed in the following subsections:

- K3 Models of multidimensional channels with fixed dimension
- K4 Models of channels with codebooks generated by cyclic sequences
- K5 Models of channels with orthogonal binary codebooks
- K6 Models of channels with suborthogonal binary codebooks
- K7 Models of channels with nonbinary codebooks
- K8 Models of channels with simplex codebooks.

It is important to notice that these various classes of models are not mutually exclusive; rather, they are grouped for reasons of convenience or historical evolution. We are still in the midst of the research period for codebooks and it is too early to recognize the most logical way or the most natural way of classifying codebook models. For an ordering of communications channels in general, the reader is referred to Shannon (58) and to T. T. Chang and Lawton (62).

K3. *Mathematical Models of Multidimensional Systems with Fixed Dimensions*

Building on Feinstein's (57) theory of memoryless channels and on Shannon's (59) theory of optimal codes in a Gaussian channel, Slepian (63) arrived at a very general model for channels with fixed dimension. We discussed Slepian's model in section D1 and we plotted his results for codes of 5, 25, or 101 dimensions as utility curves in fig. 5. One year earlier than Slepian's paper, research workers of New York University arrived at similar results using specific binary codes (S. S. L. Chang et al 62). Many of the results of this research and of related papers have recently been consolidated by Slepian (68) in his theory of group codes for the Gaussian channel. Many results of this 1968 paper are, as Slepian himself states on page 578, based on "the author's Bell Telephone Laboratories report of May 7, 1951, a document that received only limited circulation outside the laboratories".

Slepian defines *the Gaussian channel* as a model that corresponds to the codebook of l dimensions shown in fig. 16, whereby each code word (message) corresponds to a vector in an l-dimensional Euclidean space (signal space). The transmitted signal is a bandlimited version of this message vector, while the received signal consists of the sum of the sent vector and a noise vector. The components of the noise vector are independent Gaussian variates with mean zero and a variance that depends on N_o, the noise power density in the channel.

Slepian uses *equal energy block codes* of size M as codebooks. This means that the codebook has l elements and M code words, exactly as shown in fig. 16. All codewords have the same length, l, and they represent points in the vector space. The equal energy condition implies that all these vectors have their termini on a hypersphere of radius $\sqrt{lS_c}$, with S_c being the average power of the code. Slepian's theory is restricted to codebooks with $M \geq l$; it is primarily concerned with a class of equal-energy block codes called *group codes*. Although the precise definition of a group code involves mathematical expressions of the *theory of group representations* (Boerner 63), Slepian describes a group code as a codebook with important symmetry properties that cause all the code words in that group to be on equal footing. Most codes that have been investigated for the Gaussian channel are group codes; it is likely that any code used in practice will be of this type. Group codes in general require a q-ary waveform library, but the special case $q = 2$ is of high interest. All binary signalling alphabets discussed by Slepian (56) are group codes in the sense of his latest theory.

Although Slepian's theory of 1968 will prove to be of fundamental importance, the author himself concludes: "The development of this subject is clearly incomplete: We have raised more questions than we have answered". In this paper we therefore face the problem of explaining the state-of-the-art of codebook models without the benefit of an all-inclusive theory. We shall do so by converting to our terminology some of the equations that have been derived in the book by Wozencraft and Jacobs (65), henceforth to be abbreviated WJ.

The discussions of channel models with codebooks of fixed length may start with the special case $l = 2$, since the one-dimensional models have been discussed in Part II of this paper. For the two-dimensional case, C. L. Weber (65) showed that the regular M-gon is globally optimal. This result is in line with the well-known fact (Shannon 59) that the optimal choice for a two-dimensional codebook is to space the M signal vectors equally around the unit circle. An analysis of the performance of systems operating with codebooks of this kind is given by Cahn (59).

For any $l > 2$, the choice of l and M is the problem. Is it preferable to use a long codebook with only a few code words (i.e., $l > M$), a square codebook ($l = M$), or a codebook with a high and short matrix ($l < M$)?

To answer these questions we turn first to page 290 of WJ's (65) analysis of block-orthogonal signalling. There we find equation 5.12 giving an upper bound for the word error probability $P[\epsilon]$ of a mathematical model representing any set of M equally likely equal energy orthogonal signals. Converted to the notations of this paper, the upper bound for a general orthogonal model is as follows:

$$\bar{e}_w \leqslant (M - 1)\, Q \left\{\sqrt{\frac{n}{u}}\right\} \tag{56}$$

where $\bar{e}_w$ is the probability that a received code word will be in error. Let us now assume that each code word carries $n = \log_2 M$ bits of information, since this information is required for monosignal systems. The word error may then cause any number of output bits up to n to be in error. If the channel is completely symmetrical, a word error may convert the correct word with equal probability into any one of the other $M - 1$ words. In this case the word error ratio is related to the bit error ratio on the right side of equation 2, employing the equality sign:

$$2\bar{e}_B = \bar{e}_w \tag{57}$$

The Q function is the same as previously defined (fig. 8b). For the numerical evaluation we used the tables on pages 966-973 in the "Handbook of Mathematical Functions" (Abramowitz and Stegun 64), which are available for large arguments (up to 500). The utility u was defined in Part I and has been used throughout this paper.

Because we are interested in large values of M, we may make a further simplification and replace $M - 1$ by M in equation 56. This leads to:

$$\bar{e}_B < \frac{M}{2} \cdot Q \left\{\sqrt{\frac{n}{u}}\right\} \tag{58}$$

In logarithmic measures we may now express this inequality as an implicit function in u. We continue under the assumption $M = 2^n$ (monosignal systems).

$$0.30103\,(1 - n) + \log_{10} \bar{e}_B < \log_{10} Q \left\{\sqrt{\frac{n}{u}}\right\} \tag{58a}$$

To get u as a function of D—i.e., to derive the equation of a utility curve (for fixed $\bar{e}_B$ as curve parameter)—one must assume the bandwidth requirement. An idealistic assumption (leading to an upper bound for the bit density) is the maximum signal base that any orthogonal set may reach :

$$M = 2BT_B \tag{59}$$

This means that a time interval of 1/2B will be available for each dimension of the multidimensional signals or for each element of a code word in case the elements will not be further encoded into a waveform library. This ideal assumption corresponds to a transmission rate of 2W numbers or samples, i.e., to an operation at the Nyquist rate. Equation 59 can be expressed in terms of D (the bit density) and of n (the number of information bits carried by one monosignal) :

$$D^{db} = n^{db} - 3.0103\,(n-1) \tag{59a}$$

Curves can now be plotted by assuming a constant value for $\bar{e}_B$ and a set of values for n. Equation 58a yields the u value (in numbers) and equation 59a yields the D value (directly in decibels). Such curves are shown in fig. 20. Because of the approximation of 58 versus 56, the curves are unreliable for small n values.

theory must therefore investigate codebooks with fixed values of l, the number of dimensions.

Again it is Wozencraft and Jacobs who give us a convenient model for multidimensional channels with fixed dimensions, but now with a variable number of code words per dimension ($M \neq 1$). Notice that $M \gg 1$ will lead to the so-called *suborthogonal systems* (the cross-correlation coefficients between code words will be positive) while $M \ll 1$ will lead to the *hyperorthogonal systems* (systems with extremely wide bandspreading) for which the cross-correlation coefficients between code words will be negative.

WJ's model uses the Q function (i.e., the probability integral for the Gaussian distribution) and is mathematically rather complex. In fig. 2.36 on page 83 of WJ (65), one may find two approximations to the Q function, one of them using the simple Gaussian function $y = \exp(-x^2)$. It is this approximation that can easily be used to derive a conservative lower bound on the utility for all classes of codebooks and for constant error ratio. Again we take this bound from equation 5.12 on page 290 of WJ :

$$\bar{e}_w = \frac{M}{2}\exp\left(-\frac{E_s}{2N_o}\right) \tag{60}$$

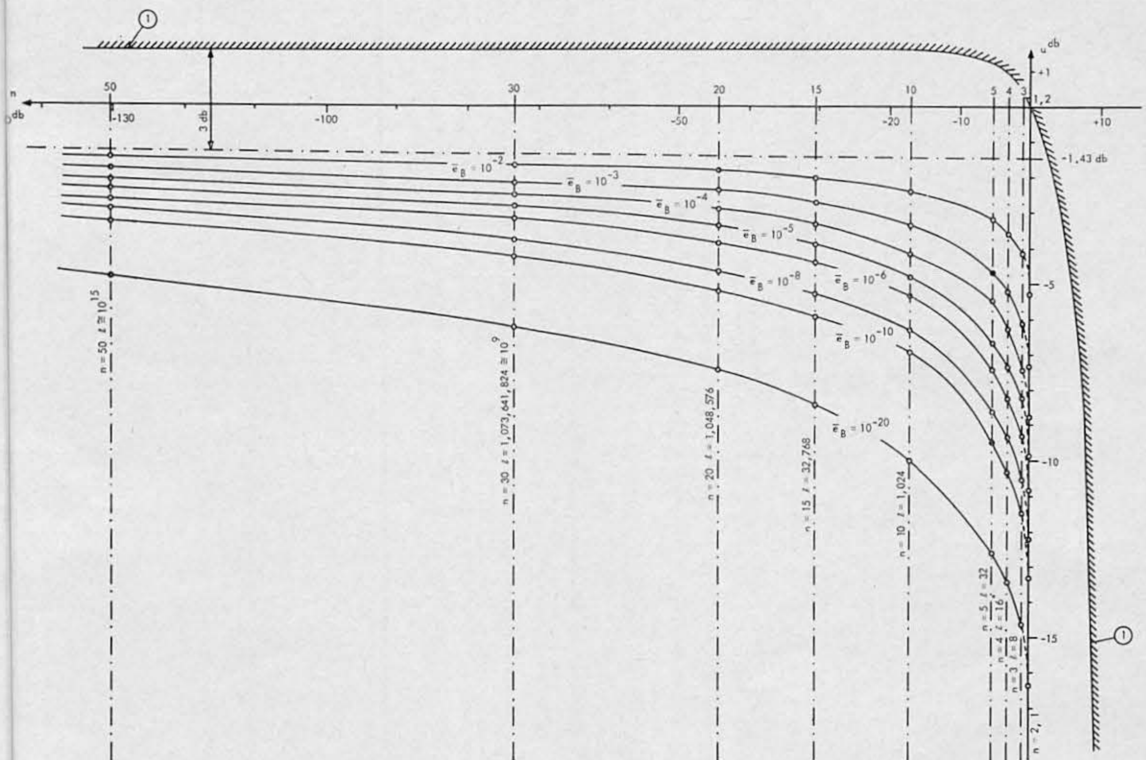


Figure 20.—Utility curves of systems with orthogonal binary codebooks.

The codebook with $M > 1$ (vertical rectangle) will be more desirable in most applications than the codebook with $M < 1$ (horizontal rectangle). The reason is that the bit density can not go below something like −50 or −60 db because the bandwidth requirement becomes excessive for even very moderate information transmission rates. If one desires to increase the utility without decreasing the bit density, one can do so only by increasing both the length of the code (l) and the number of code words. In monosignal transmission the number of code words increases exponentially (to the base 2) with the length of the code word, to keep the ratio of l/n (and therefore the bit density) constant. Thus it is clear that the codebooks will result in higher and higher rectangles ($M \gg 1$), when trying to improve the utility without changing the bit density. The

Wozencraft and Jacobs show how this bound can be applied to the special case of binary elements (q = 2 in fig. 16). Assuming further that the two waveforms of this degenerate waveform library will be antipodal to each other, WJ arrive, in equation 5.38c on page 203, at the following mathematical model for systems with *codebooks consisting of binary elements*, written in WJ notation :

$$\overline{P[\epsilon]} < 2^{-N[R_o - R_N]} \tag{61}$$

In this notation, $\overline{P[\epsilon]}$ corresponds to the word error ratio ($\bar{e}_w$) of this paper. N is the number of dimensions corresponding to l ; R_N is the transmitter rate in bits per dimension, corresponding to n/l ; and R_o is called the *exponential bound parameter*. R_o establishes a limit for improvement.

As long as $R_N < R_o$, the word error ratio, $\bar{e}_w$, can be made arbitrarily small by taking l sufficiently large. R_o can be expressed in terms of the magnitudes used in this paper by using WJ equation 5.36b on page 303, and when introducing β_c as the signal base required for one dimension. $\beta_c = 0.5$ is the minimum value for β_c (Nyquist rate):

$$R_o = 1 - \log_2 \left[1 + \exp\left(- \frac{\beta_c D}{u} \right) \right] \tag{62}$$

In this equation, D is the bit density (equation 11) and u is the utility (equations 13 to 15). Combining equations 61 and 62, inserting the magnitudes used in this paper, and replacing the unequal sign by an equal sign lead to the following equation for a very conservative upper bound of the utility of an ideal monosignal system with a codebook of $M = 2^n$ code words and a code length of $l = n/\beta_c D$:

$$u = - \frac{\beta_c D}{\ln (2^L - 1)} \tag{63}$$

$$u^{db} = \beta_c^{db} + D^{db} - 10 \log_{10} [-\ln (2^L - 1)] \tag{63a}$$

The constant L is defined by the equation:

$$L = \frac{\beta_c D}{n} (\log_2 \bar{e}_B + 1) - \beta_c D + 1 \tag{64}$$

Figure 21 shows several curves following equation 63 with the parameters $\bar{e}_B = \bar{e}_w/2 = 10^{-3}$ (solid lines) and $\bar{e}_B = 10^{-6}$ (dot-dash lines), and with parameters n = 3, 5, 10, 20, 30, 50. Curves of constant l values are shown as dashed lines for $\bar{e}_B = 10^{-3}$ only. They do not apply for the lower error ratio of 10^{-6}. All the curves of fig. 21 are plotted for the minimum character signal base $\beta_c = 0.5$ (Nyquist rate).

For interpreting the meaning of fig. 21, assume that some one designed an ideal conceptual system with a binary codebook of l = 40, $M = 2^{10} = 1024$, operating at the Nyquist rate of two dimensions per Hertz bandwidth and using perfect maximum likelihood detection in a Gaussian channel. The designer can then anticipate operating with an error probability of one bit in one thousand at the output of his system. This is characterized by the operating point P in fig. 21.

There are several important aspects that should be remembered when using a utility chart of an ideal mathematical model such as fig. 21, and when selecting an operating point such as P:

(i) The mathematical model always makes idealized assumptions that cannot be completely realized in engineering implementations. In all the models mentioned in this section we assume distortion-free circuits, absolutely perfect synchronization, perfectly generated signals that correspond exactly to their mathematical definition, etc.

(ii) The mathematical model may have been defined purposely for a worst limiting case. The models of figs. 20, 21, and 22 use all the worst possibility when converting word errors into bit errors. We assume that on the average, half of the bits of an output word will be in error if the word itself is in error.

(iii) The mathematical model will usually be valid only for certain statistical characteristics of the input parameters. In all our models we assume that all possible input messages will occur with equal probability.

(iv) The mathematical model may be representative for a class of systems and not for a single special system. This is the case for the models represented by figs. 21 and 22. They actually represent an average of all possible binary codebooks, as explained on pages 298-307 of WJ (65). Some codebook configurations may therefore yield better results; others will yield worse results. More restrictive models will be needed to describe the ideal performance of any special codebook configuration.

Having all the above limitations in mind, one can still derive great benefits from the model represented by fig. 21. It is also interesting to note that, when keeping the code length constant (for example, l = 40) and when increasing the number of code words considerably (for example to 2^{24}), one can find another operating point for the same utility (−5.3 db) but at much higher bit density (D = +0.7 db).

On the other hand, when the bandwidth requirement for a

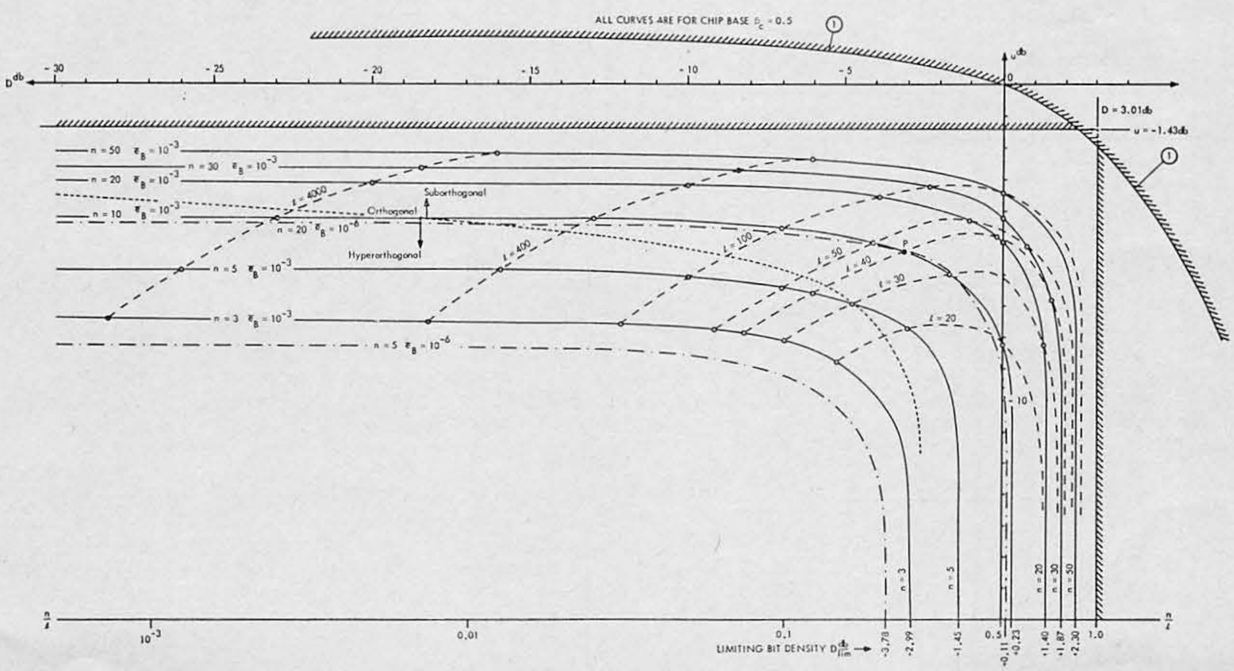


Figure 21.—Utility curves of systems with binary codebooks of various dimensions.

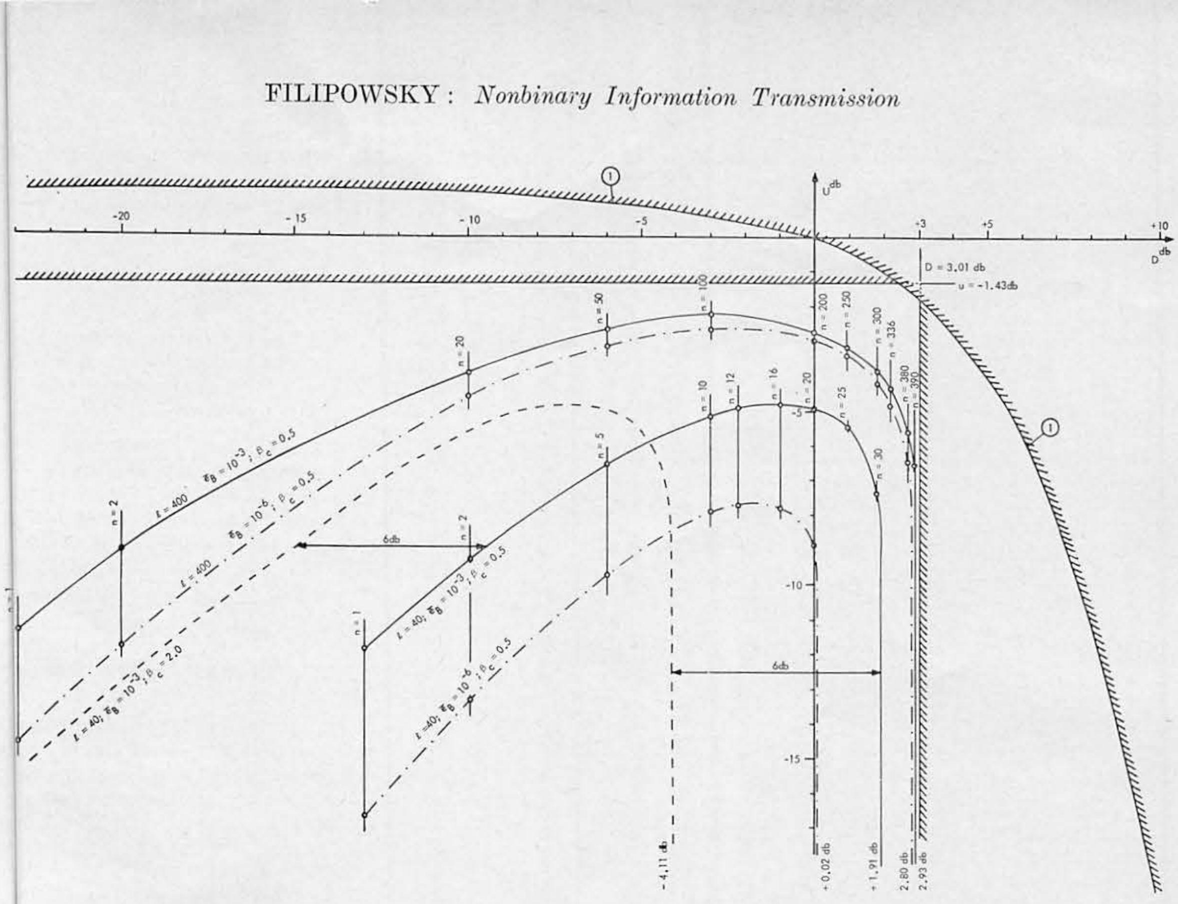


Figure 22.—Comparison of the utility curves of systems with codebooks of 40 and 400 dimensions.

design is not critical, i.e., when the designer can afford to go to much lower bit density, it is possible to enlarge the codebook by increasing the length of the codebook but keeping the number of codebook words constant. In this case one moves on a solid line to the left. For example, when going from P on the line n = 10 to the left, one can see that increasing the length of l from 40 to 100 can cause a gain in utility of about 0.7 db. The bit density is then reduced to −7 db. A square codebook will be reached at about −18 db bit density. Here an orthogonal codebook may be used with a utility of −4.2 db and a SNR of −13.8 db. Notice, however, that the model of fig. 20 needs only −3.5 db utility for the same bit density of −18 db. The difference is explainable by the fact that fig. 21 is restricted to binary sequences and is an average of all possible codes.

The most important results of fig. 21 are the maxima of the dashed lines. They indicate that, for fixed code length l and for given error probability, there is a maximum utility that will be reached at a redundancy between 1/4 and 1/2. This maximum of the utility is several decibels higher than the utility that can be reached by an orthogonal codebook at a similar bit density. It offers operating points at a much higher bit density than the operating points for orthogonal codebooks, when making the comparison for constant utility and constant error ratio. To express these facts still better we derive the equations for the utility curves for codebooks of constant length. In analogy to equations 63 and 64 we can write:

$$u = -\frac{\beta_c D}{\ln\left(2^{L' - \beta_c D} - 1\right)} \tag{65}$$

$$u^{db} = \beta_c^{db} + D^{db} - 10 \log_{10}\left[-\ln\left(2^{L' - \beta_c D} - 1\right)\right] \tag{65a}$$

$$\text{with} \quad L' = \frac{1}{l}\left(\log \bar{e}_B + 1\right) + 1 \tag{66}$$

The curves are plotted in fig. 22 for $\bar{e}_B = 10^{-3}$ as solid lines and for $\bar{e}_B = 10^{-6}$ as dot-dash lines. Only two parameters are used for the code length: $l = 40$ and $l = 400$. A fifth curve is added to show how a different value for β_c will affect the curve. The dashed curve is plotted for $\beta_c = 2.0$, a rather comfortable signal base for one dimension (chip). Note that an increase of β_c shifts the curve merely parallel to itself to the left, exactly for the amount in decibels that β_c increased (6 db in this case).

Figure 22 shows some interesting characteristics for systems with suborthogonal block coding. At the left end of each curve the codebook has only two code words (n = 1). Notice that these points are far below the utility lines of the ideal binary system, though the choice of two antipodal binary sequences as only two waveforms of an ideal binary system would correspond completely to the mathematical model of equation 34. But we must remember that the model of fig. 22 represents the *average* of all possible codes and that there are many other pairs of binary sequences besides the optimal antipodal sequences.

As one progresses to the right on any curve of fig. 22, one deals with codebooks of the same length l and of increasing numbers of code words. When $2^n = l$, one reaches the point where orthogonal binary sequences may be used (square codebooks). Again it should be noted that codebooks with orthogonal sequences are a very special case of square codebooks and it is to be expected that the average square codebook will offer a lower utility than the optimal orthogonal codebooks can offer.

Increasing n over the value $\log_2 l$ leads to suborthogonal codebooks and to a decreasing redundancy. The codebooks now become very high rectangles. At the points of maximum utility one has codebooks with $2^{l/2}$ and more code words. We recognize that to reach the attractively high utility of −2.2 db, the codebook would need $2^{100} > 10^{10}$ different code words, even though the code length of 400 binary elements would be manageable. Beyond the maximum of utility, the number of code words increases further and fast approaches the maximum number of 2^l code words of a complete codebook. Before this ultimate binary rate can

be reached, the curves turn towards $u^{db} = -\infty$. This is the limit where, in equation 61, $R_N = R_o$, the exponential bound parameter. The following equations give the limiting values directly:

$$D < \frac{1}{\beta_c\, l}(\log_2 \bar{e}_B + 1) + \frac{1}{\beta_c} \tag{67}$$

Letting l, the codebook length, grow beyond all limits pushes this limit for D to the value $-\beta_c{}^{db}$. If β_c is at the Nyquist rate ($\beta_c = 0.5$; $\beta_c{}^{db} = -3$ db), we see that the D limit moves also to the Nyquist rate ($D^{db} = 3$ db). If $\beta_c = 2$, the D limit must be left of the -3 db line.

Corresponding to the D limit the limit for n—if l and $\bar{e}_B$ are given—is:

$$n < \log_2 \bar{e}_B + 1 + l \tag{68}$$

There is also an upper limit for the utility of multidimensional systems with binary sequences as codebook words. This limit is derived on page 290 in Wozencraft and Jacobs (65) from the exponential bound. It is given by:

$$u < \frac{1}{2\ln 2} = \frac{1}{1.39} \quad \text{or} \quad u^{db} < -1.41 \text{ db} \tag{69}$$

Kurz (61) developed a theory of *efficient codes*, including minimax, equal separation, and nearly equal separation codes. The codes are formed from weighted sums of eigenfunctions generated by an integral equation with its kernel corresponding to the inverse Fourier transform of the Gaussian noise spectrum. This theory accomplished the extension of a general mathematical model of multidimensional nonbinary systems to channels with nonwhite Gaussian noise. Wolfowitz (60) initiated the concept of a channel where neither the sender nor the receiver knows the channel probability function, i.e., the error probability for given messages. Based on this idea, Wolfowitz provided in 1961 "an easy introduction to the ideas and principal known theorems of a certain body of coding theory for mathematicians of some maturity" (Wolfowitz 61). Though the author claims in the preface that "the book has no pretension to exhaustiveness, and, indeed, no pretensions at all", any serious student of the evolution of information theory in the 1950's can use this book as a sheer inexhaustible source of advanced proofs and theorems, enabling him to put the engineering ideas of the 1960's on a solid theoretical basis.

Another book by Gallager (63) concentrates on the search for efficient binary codebooks by consolidating the research of the 1950's in this area. Gallager deals with a special class of parity check codes, called low-density parity-check codes. These codes are characterized by codebook matrices containing mostly zeros and relatively few ones. Chapter 5 of the book extends the major results of the other chapters to nonbinary, low-density parity-check codes. Beyond the treatment of these special low-density parity-check codes, Gallager (65) offers a simple derivation of the coding theorem that gave us the exponential bound of the error probability in the form that we used for figs. 21 and 22.

The optimality of the simplex code and several other codes in achieving, over a Gaussian channel, the absolutely highest utility (if bandwidth requirements are of no concern) was put on a solid theoretical basis by Landau and Slepian (66). The previously mentioned summary report by Gibson (66) may be welcome to those readers who are familiar with the communications efficiency as defined in Sanders (60a). Gibson's summary compares multidimensional nonbinary communications systems with one-dimensional nonbinary (M-ary) systems.

K4. *Mathematical Models of Systems with Codebooks Generated from Cyclic Sequences (Shift Register Sequences)*

We shall discuss the experimental communications systems that operate with binary shift register sequences in section L. Most of the theoretical work in this area is still based on the original contributions of Reed (54), describing the famous Reed-Muller codes, and on the two original investigations of shift register sequences by Zierler (55) and Golomb (55). A later book (Golomb 64) gives some more recent results. Basically all the models resulting from these sources are well in line with the general models discussed above. Shift register sequences and, in particular, the so-called pseudonoise (PN) or pseudorandom (PR) sequences are also used in asynchronous multiplex systems.

K5. *Mathematical Models of Systems with Binary Orthogonal Codebooks*

Equation 56 was used as a convenient mathematical model for any discrete data transmission system that uses a set of M equally likely equal energy orthogonal signals. Equation 58, a further simplification of equation 56, has been plotted in fig. 20 over a wide range of bit densities and for many error probabilities.

In this subsection we shall discuss more precise mathematical models, many of which have been derived for specialized channel conditions. We intend to follow the historical evolution, though we take the liberty of discussing the works of one and the same author out of historical order.

The oldest mathematical model, as far as this author could establish, is contained in Kotel'nikov (59a), originally published in 1947. With reference to the English edition, we notice equation 5-12 in Kotel'nikov's notation:

$$P(A_1 = \text{correct}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[1 - V\left(\frac{\sqrt{2Q}}{\sigma} + y\right)\right]^{M-1} \exp\left(\frac{y^2}{2}\right)dy$$

In this equation Kotel'nikov expresses the probability of correct reception if $A_1(t)$ is being transmitted and the receiver makes a maximum likelihood decision. $A_1(t)$ is one waveform out of an orthogonal set of M equally probable equal energy waveforms. All waveforms are defined over a time interval of T seconds and are zero outside this interval. Accordingly P is the probability of correct reception for a communications system operating with an orthogonal set of waveforms. Kotel'nikov uses in his model the following notations:

p. 11: $$V = \frac{1}{2\pi}\int_{-\infty}^{-X} e^{-z^2/z}\,dz \equiv Q\{x\}$$
(See equation 27 in this paper.)

p. 30: $$Q^2 = \int_{-T/2}^{+T/2} A_1^2(t)\,dt \equiv E_D$$
energy of one waveform (digit).

pp. 10-15: $\sigma^2 \equiv 2N_o$ — N_o . . . noise power density.

$P \equiv 1 - \bar{e}_D$ — $\bar{e}_D$ . . . average error probability per waveform.

z is a dummy variable for the integration process.

(The page numbers and the notations at the left side of the identity sign ($\equiv$) in the above equations refer to Kotel'nikov's book.)

Many other authors, independent of Kotel'nikov, arrived at the same expression for the probability of a correct decision in an orthogonal system. We mentioned Nuttall's (62) results in connection with our equation 54, which for $\lambda = 0$ is identical with Kotel'nikov's expression. Reiger (57) published an equation that proves to be identical with Kotel'nikov's expression when applying a small substitution of the dummy integration variable. Viterbi, who derived the same equation in various publications since 1960, gives a table of the error probabilities (Viterbi, 64b, Appendix 4) that can be calculated from Kotel'nikov's expression.

When transforming Kotel'nikov's model to the definitions

of the utility chart, one must establish a relationship between the digital error probability $\bar{e}_D$ and the binary error probability $\bar{e}_B$. For the curves of fig. 20, we used equation 57 for this relation. A more precise relationship has been derived by Viterbi (66) and by other authors (equation 70):

$$\bar{e}_B = \frac{M}{2(M-1)} \cdot \bar{e}_D = \frac{2^{n-1}}{2^n - 1}\bar{e}_D \tag{70}$$

$M = 2^n$ is the number of waveforms participating in the orthogonal monosignal system that carries n bits of information in each transmission interval.

Before expressing Kotel'nikov's equation in the magnitudes of the utility chart, one must take the energy with reference to one information bit; i.e., $E_D = nE_B$. This leads to the implicit equation for the utility $u = N_o/E_B$:

$$\bar{e}_B = \frac{2^{n-1}}{2^n - 1}\left[1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-y^2/2}\left[1 - Q\left\{y + \sqrt{\frac{n}{u}}\right\}\right]^{2^n-1} dy\right] \tag{71}$$

From equation 27 it is shown that:

$$1 - Q\{x\} = P\{x\}\,;\ P\{x\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}\,dz$$

With the help of this relation, one can express the integral in equation 71 in the form of equation 54 and use the tables in one of the references quoted in subsection J7. Viterbi (64, Appendix 4) has calculated $\bar{e}_B$ and $\bar{e}_D$ as a function of $ST_B/N_o = 1/u$ with n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, and 20 as parameters. Some of these curves are shown in fig. 2 of Part I of this paper (in decibel scale).

When plotting a utility chart one requires a second equation that relates the parameter n (bits per digit or word) to the bit density D. In fig. 2 we used a sliding relationship between the binary error ratio $\bar{e}_B$ and the signal base per dimension (chips or binary sequence element) β_C. In figs. 20, 21, and 22 we used a constant value of $\beta_C = 0.5$ (Nyquist rate) for all curves except one for which $\beta_C = 2.0$ had been selected. Likewise in section 8.5 of his 1966 book, Viterbi, in line with the tradition of the codebook school, recommends the use of the minimum signal base $\beta_C = 0.5$. However, he points out the fallacies of the use of sin x over x waveforms that had been recommended by other authors as a model for the signal shape when postulating a signal base of $\beta_C = 0.5$. Instead, Viterbi recommends rectangular envelopes of sine and cosine functions. He defines as bandwidth not the 3-db bandwidth or any other limitation of the essential width of the power spectrum but the " minimum frequency separation between channels such that signals of any channel have no effect on the decoders of the others ". To demonstrate the practicality of this idea, Viterbi describes the conceptual model of a " coded phase-coherent communications system " that uses orthogonal binary sequences as orthogonal signal sets with each binary chip being represented by one of two phase states of a sinusoidal carrier frequency. The two states have a phase difference of 180 degrees (Viterbi 60a, b). Thus we see that the waveform library degenerates in the coded phase-coherent communication system to a binary biphase alphabet (section H). If one would really build a system of this kind with many densely packed subcarriers so that Viterbi's definition of bandwidth could be realized, one would arrive at a polysignal system similar to the well-known Kineplex data transmission system.

Accepting the $\beta_C = 0.5$ value for the sake of defining the most efficient ideal model that can be defined for binary chips without mutual interference, we arrive at the second equation for systems with orthogonal binary sequences:

$$D = \frac{R_B}{B} = \frac{R_B T_C}{\beta_C} = \frac{R_B T_C}{0.5} = \frac{n}{2^{n-1}} \tag{72}$$

This equation is identical with equation 59a, which is written in decibel notation. Any curve $u = f(D)$ for constant $\bar{e}_B$ and for n as the running parameter, when calculated with the help of equations 71 and 72, can therefore be directly plotted into charts 20, 21, or 22 and can be compared with the other mathematical models contained in those charts. When doing so one will find that the Kotel'nikov-Viterbi model is more accurate and arrives at higher utilities than the oversimplified model of equation 58a and fig. 20. The model of fig. 20 shows the slow gain in utility even when going to such extreme values as $n = 50$ and $\bar{e}_B = 10^{-20}$, values that are not covered by Viterbi's charts or tables.

Kotel'nikov's model of 1947 and Viterbi's subsequent implementation with the help of binary sequences are not the only attempts to use orthogonal signal sets for the purpose of maximizing the system utility. As early as 1949 Golay proved that an extremely simple model of an orthogonal system can reach the highest possible utility for the Gaussian channel. Golay (49b) showed that, in a system that transmits in one-out-of-M timeslots, one pulse of energy E_D can reach the upper limit of the utility $u = (\log_e 2)^{-1} = 1.441$ (or 1.59 db) when M goes toward infinity. A system of this kind is called a quantized pulse position modulation system (QPPM) or a multiple time shift keying system (MTSK). Systems of this kind will be mentioned in section L3.

Similarly it was shown by Turin (59) that the Kotel'nikov model in Reiger's form (57) can be used to submit the general proof that all orthogonal monosignal systems ultimately reach the upper limit of the utility when M grows beyond all limits.

Continuing in the chronological discussion of mathematical models for orthogonal systems, we find that Harmuth published in 1960 three papers with detailed analyses of the transmission of information by orthogonal time functions (Harmuth 60a, b, c). Two different kinds of orthogonal sets are discussed:

(i) sets consisting of binary words

(ii) sets consisting of trigonometric waveforms

The first class of sets uses rectangular signals in the time domain and displays sin x over x frequency spectra in a low-pass channel. The second class of signals uses rectangular envelopes on sinusoidal subcarriers. When used in a low-pass channel, the spectrum of such waveforms is much narrower than the channel bandwidth and is centred at the frequency of the particular subcarrier. These spectra also have sin x over x shape. When such signals of the second class are used in a radio channel (bandpass channel), the subcarrier frequencies may be directly placed into the carrier band by single sideband modulation, or the subcarrier frequencies may be formed in the low-pass channel and go by double sideband modulation into the carrier band. In this case they form a symmetrical spectrum at both sides of the radio carrier frequency, and the total bandwidth will be doubled. In all the cases mentioned above, there is a small amount of cross talk generated between the various waveforms when the channel must be exactly bandlimited. The amount of such cross talk is computed in Harmuth (60b). The error probability of the systems is discussed in all the papers, but in one paper (Harmuth 60c) an explicit formula is derived for the error probability when the number of orthogonal functions is very large. A special codebook of 32 binary orthogonal code words is disclosed (Harmuth 60c) and a special encoding and decoding procedure by matrix multiplication is explained (Harmuth 60).

Posner (62) succeeded in investigating the performance of orthogonal systems with correlation detection at very low SNR. He uses a mathematical approximation to equation 71 that is valid for small SNR and that may be of interest to some readers. With the help of this approximation, Posner showed that at " output SNR of less than one " (corresponding to utilities in the positive decibel range)

orthogonal codes may be less efficient than no coding at all. So far this author has had no opportunity to convert Posner's results to the notations of the utility chart, but the reader is invited to verify that the ideal binary system (equation 34) has a utility of +0.87 db ($u = 1.22$) for a bit error ratio of $\bar{e}_B = 10^{-1}$ while Viterbi's tables (Viterbi 64b, Appendix 4) indicate that an orthogonal system needs at least $n = 4$ ($K = 4$ in Viterbi's tables) to reach $\bar{e}_B = 10^{-1}$ for $u = 1.22$ ($ST_B/N_o = 0.82$ in Viterbi's table). This means that, for an error ratio of 0.1, orthogonal codes seem to be inferior to a simple binary system (both operating with correlation detection) when the code length l is smaller than 16 chips. At such large error ratios, it is more profitable to use two antipodal groups of 4 chips as binary biorthogonal waveforms than to use a codebook of 16 chips and 16 code words. Apparently, for utilities larger than +0.87 db, the situation will be much more in favour of uncoded transmission. Posner's investigation produced four theorems that may prove helpful in any research for transmission of digital information in channels with extremely small energy contrast (SNR).

All the models discussed so far assumed a perfect cross-correlator, completely coherent reception, and perfect synchronization. These are rather ideal assumptions; practical systems operating over channels with fading, or between moving stations are not able to meet these assumptions. Less ideal models were therefore required and the first step towards more realistic mathematical models were those that assume no phase coherence. Ward (62) submitted a formula for the error probability of a receiving system with l independent channels, where only one signal is expected in one of the channels and the channel with the signal has to be detected by a " largest of " selection. The signal is immersed in Gaussian noise, all channels are assumed to have the same noise power density N_o, and all channels employ envelope detectors. This model can correspond to a very simple orthogonal system where either l narrow band channels are available and a signal of energy E_D is transmitted in one of them (MFSK), or where l time slots are available and a signal of energy E_D will be transmitted in only one of them (MTSK).

Dumanian (63) expanded the line of thinking that had been expressed by Ward (62) with a formula for the error probability in the case where the receiver uses a correlator after the envelope detector and before the " largest-of " selection. Sommer (63) had independently gained results similar to those reported by Ward (62), and Splitt (63) derived an expression for the limiting form of the probability of error as the number of channels or slots (M) approach infinity. This communication by Splitt asserts that, also in the noncoherent case (envelope detector), the upper limit of the utility $u = (\log_e 2)^{-1}$ will be reached for $M \to \infty$ ($D \to 0$). Teplyakov (63) derived an approximation of Kotel'nikov's integral expression (equation 71). This approximation contains only the error integral (fig. 8b) and no further integration process. It is valid for very large values of M.

Lindsey (65a) arrives at the most universal model for systems with sets of orthogonal signals. Lindsey's model can be applied to the noncoherent case, to the partially coherent case, or to the coherent case. The latter specialization leads to the earlier results of Kotel'nikov (59a), Reiger (57), Turin (59), and Nuttall (62). Lindsey (67) expanded his results to the polysignal case, where several independently phase-modulated carriers go over the same radio carriers.

Biorthogonal systems are a simple extension of the orthogonal systems when applied in connection with binary sequences. They use a rectangular codebook of l chips but 2l words. Half the number of words is a normal orthogonal codebook; the words in the second half are exactly the negative of the words in the first half. Thus each word in a biorthogonal codebook has one antipodal word and 2l-2 orthogonal words. The idea to extend an orthogonal set to a biorthogonal set is so evident that it is difficult to credit any one of the many research workers in this area with the origination of biorthogonal systems. In the extreme case of $l = 1$, a biorthogonal system with correlation detection becomes identical with the ideal binary system.

The first biorthogonal codebooks are due to Muller (53) and Reed (54); a biorthogonal system was derived and subjected to computer evaluation by Viterbi in 1960. Also Sanders (59a) and Harmuth (60c) independently proposed the application of biorthogonal codebooks in data transmission systems. A very general proof of the optimality of biorthogonal codebooks is due to Balakrishnan and Taber (63).

Strong and Saliga (65, 66) have recently reviewed most of the above-mentioned mathematical models. They demonstrate in a number of charts and tables, that for large codebooks ($n > 6$), phase-coherent orthogonal systems have a better than 3-db advantage in utility over noncoded systems. This advantage increases to more than 4 db when $n \geq 10$ ($M = 2^{10} = 1024$). The utility advantage of a biorthogonal system is practically the same as the one for an orthogonal system for all $n > 3$. At $n = 3$ and below, biorthogonal systems offer a slightly higher utility than phase-coherent orthogonal systems. At $n = 2$ and $n = 1$, the biorthogonal system is identical to an uncoded system, indicating that any kind of orthogonal coding is inefficient at short codebook lengths.

When comparing phase-coherent orthogonal systems with non-phase-coherent orthogonal systems (such as MFSK), Strong and Saliga show that, for large n values, the utility of the non-phase-coherent system is only less than one decibel below the utility of the phase-coherent system. This advantage of the phase-coherent system increases to about 2 db at $n = 3$ and at a word error probability of 0.1 (corresponding to $\bar{e}_B = 0.057$, according to equation 70). But notice that at high word error probabilities the disadvantage is always smaller than at lower error probabilities, no matter what n value has been selected.

Strong and Saliga do not discuss the bandwidth requirements in detail but they note that the biorthogonal system needs only half the bandwidth of the phase-coherent orthogonal system. Also interesting in their charts is the fact that the uncoded system seems to have higher utility at very low error probabilities (corresponding to the highest utility values or to the lowest SNR values in units of E/N_o = energy contrast). This would support the calculations of Posner (62) that were mentioned above. Unfortunately Strong and Saliga plot their curves only for $\bar{e}_w < 10^{-1}$. For $n = 10$ this corresponds to $\bar{e}_B < 0.05$; thus they avoid the range for which Posner proved the superiority of uncoded transmission.

C. L. Weber (66) continued Balakrishnan's research (65) and demonstrated that, for a given SNR (*utility* in the terminology of this paper) and for a given allowed bandwidth (bit density), the problem of finding the optimum set of signals can be reduced to the problem of finding a non-negative definite matrix that maximizes the probability of detection. For a rectangular codebook matrix with $M = 2l$, the biorthogonal signal structure achieves the maximum probability of detection at large SNR. The same author develops with R. A. Scholtz as co-author (Weber and Scholtz 66) a further extension of this theory to the case of non-phase-coherent communications, showing that the orthogonal set is yielding the lowest probability of error in the class of all admissible signal sets and that this result is true for all signal-to-noise ratios. This result could be interpreted as indicating a contradiction of Posner's results, who proved that uncoded transmission is superior to coded transmission at low SNR. Notice, however, that Posner compares the systems for phase-coherent reception while Scholtz and Weber speak exclusively of non-phase-coherent communications. Also, Posner searches for the system with the highest utility for a given error probability, while Scholtz and Weber search for the system with the lowest error probability for a given utility.

K6. *Mathematical Models of Systems with Suborthogonal Codebooks*

Suborthogonal codebooks are those where some or all pairs of code words have a positive cross-correlation coefficient. Even under noise-free conditions, this will cause some positive output from correlators of waveforms that have not been transmitted. If the noise power is correspondingly smaller than in the orthogonal case, correct MLD decisions are nevertheless possible with the same probability as in the orthogonal case. Systems with suborthogonal codebooks are therefore useful where operation with lower utility but higher bit density is the goal. Fig. 21 shows the area of suborthogonal systems for the very general mathematical model of equation 63. We recall that said model applies to the average of all conceivable binary codebooks of the given ratio $2^n/l$.

Helstrom (55) derived the famous equation for ideal binary systems (equation 34) :

$$e_B = Q\left\{\sqrt{\frac{1-\lambda}{u}}\right\}$$

λ is the cross-correlation coefficient between the two binary waveforms. If $\lambda = 0$, we have the orthogonal case ; for $\lambda > 0$, the system is suborthogonal ; for $\lambda < 0$, it is hyper-orthogonal ; and, for $\lambda = -1$, biorthogonal or antipodal. We recall that this equation applies to any pair of waveforms ; therefore, it is also valid for any two waveforms consisting of sequences of binary elements. The cross-correlation coefficient is in this case the *ratio of* the difference between the number of elements in agreement and the number of elements in disagreement (between the two words) *to the* total number of elements. Such a codebook consisting of only two words can be represented by a matrix of l columns and 2 rows.

Slepian (56) systematically investigated all binary codebooks up to $l = 10$ as a generalization of Hamming's error-correcting codes. Slepian defined a class of codebooks that he called group alphabets. This class included Reed-Müller codes (already mentioned as the basis of biorthogonal codes) and systematic codes as originally defined by Hamming (50). Most of Slepian's binary signalling alphabets are suborthogonal codebooks and qualify completely as nonbinary systems when used in connection with MLD decoders. Although Slepian establishes important theorems for finding " best group alphabets ", i.e., suborthogonal codebooks that maximize the probability of correct detection with an MLD decoder, he does not give an explicit or implicit formula relating the error probability to the SNR (utility).

After research efforts had addressed the more fundamental problems of finding orthogonal, biorthogonal, and simplex codebooks, interest returned in the last years to systems with suborthogonal codebooks. (See, for example, Viterbi 66, chapters 8.2 and 8.8, and Kurz 61.)

Hackett (63) published a theory for systems with rectangular codebooks (of binary elements) and compared the various decoding methods.

C. L. Weber (66) found " new solutions to the signal design problem for coherent channels " by applying a new concept to the design of codebooks with $M > 1$. He demonstrated that it is possible to place K lower-dimensional regular simplices mutually orthogonal to one another to arrive at optimal suborthogonal codebooks.

When continuing to search for the mathematical models of more specific suborthogonal codebooks, we notice that Bose-Chaudhuri (BC) codes are used for the most preferred suborthogonal codebooks (Gibson 66). These codes were first published in 1960 (Bose and Ray-Chaudhuri 60) and Van Horn (61) developed a mathematical model for the error probability when using these codes with bit-by-bit decoding procedures. The results of Van Horn's investigation have recently been compared in a utility chart with the general model which uses the MLD procedure (Filipowsky 68b). Some utility curves of this MLD model are shown in fig. 22. The comparison in Filipowsky (68b) indicates that BC codes should be very attractive candidates for suborthogonal block coding with MLD. An interesting compromise between bit-by-bit decoding and MLD (also known as probabilistic decoding) is called " generalized minimum distance decoding " and has recently been introduced by Forney (66).

A number of other suborthogonal codebook models have become known but the respective publications do not offer any new mathematical systems models.

K7. *Mathematical Models of Systems with Nonbinary Codebooks*

Shannon (48) ; Feinstein (54), (55), and (58), Fano (52) and (61) ; and Wolfowitz (57) are the principal contributors to a group of mathematical models of communications systems that are now generally known as the exponential error bounds (WJ 65). These error bounds are likewise applicable to systems with binary codebooks and with nonbinary codebooks. When discussing the various classes of binary codebooks, we used these bounds with good results. For nonbinary codebooks they are the backbone for all mathematical models.

Wolfowitz (61) gives on pages 30-32 a short history of the evolution of the exponential error bounds, stressing the fact that this whole approach has been " brilliantly conjectured " by Shannon in 1948. In that 1948 paper and still more explicitly in 1959, Shannon stressed the fact that the symbols for the code words (the letters in the codebook matrix of fig. 16 in this paper) must be taken from a large alphabet (our waveform library). Ideally, the size of the alphabet should be infinite, permitting all real numbers to be used as letters for the codebook. Shannon and most other authors also invariably use an MLD message (word) decoder as the optimum receiver when deriving their mathematical models for nonbinary codebooks. Notice also that most authors follow Shannon's example and use for their models the minimum theoretical signal base $\beta_c = 0.5$ (Nyquist rate). Practical systems require a larger signal base (time bandwidth product per letter or per dimension of the codebook) than do theoretical systems.

Shannon (59) lists three possible limitations on the choice of code words in a codebook with nonbinary letters. These limitations lead to three classes of codebooks that yield quite similar results in their overall performance but still require different equations for their mathematical models. Shannon also stresses the fact that he deals in his 1959 paper only with the case of average power limitation on the code words, whereas in his earlier paper in 1948 he discussed also the channel capacity with a peak power limitation. The three classes under average power limitation discussed in the 1959 paper are the following :

(1) All code words are required to have exactly the same power P or the same distance from the origin (in an l-dimension vector space). Code words are in this case represented by vectors with end points lying on the surface of a hypersphere of radius $\sqrt{lP}$.

(2) All code words have power P *or less*. In this case all code words lie on the surface of a hypersphere or within its interior.

(3) The *average power* of all code words is P or less. That means some code words may lie outside the sphere of radius $\sqrt{lP}$ and others may lie inside the sphere. If all code words are used equally likely, the average power will remain P ; but, if some code words have a different probability from others, the average power over a long interval may differ from P but never should be larger than P.

In the following conversions of the results of Shannon and of other authors to the notations of the present paper, we shall always assume codebooks of class (1) unless stated differently.

The power of a codeword is defined as :

$$P = \frac{\sum_{i=1}^{l} \beta^2{}_i}{l} \qquad \text{in } \beta^2 \text{ units} \tag{73}$$

for a code word of l real numbers β_i (fig. 16).

The exponential bounds, in their most general form, may be expressed in the following form :

$$\bar{e}_w = C(l, D) \,.\, \exp\{-l \,.\, E_L(l, D)\} \tag{74}$$

The various forms of exponential bounds differ in the structure of the two functions C(l, D) and E_L (l, D). The function C(l, D) is frequently called the " constant " of a particular exponential bound. It is true that its influence on the error probability is less significant than the influence of E_L (l, D). This second function E_L (l, D) is normally called the reliability function of any exponential bound. Indeed, the " constant " may be represented by a constant value over large ranges of l and D. For the sake of simplicity we shall normalize this " constant " to unity for much of the following presentations. The reader will easily recognize how to introduce other values of C(l, D), when required for special purposes. Shannon (59) and Gallager (65) offer many equations for C(l, D), thus permitting the application of the normalized exponential bounds to many special cases. Brillouin (56) relates this general exponential bound (his equation 17.16) to the general probability law of thermodynamics.

The *reliability function* E_L (l, D) determines the general behaviour of the mathematical models of codebooks with nonbinary elements. Shannon (59) offers a series of charts for E_L (R_{SH}) which can be used easily to derive utility curves for mathematical models of nonbinary codebooks. Figure 23 shows how to use Shannon's charts. At the left side there is a typical set of curves showing Shannon's E_L for the case of his parameter A = 2 (taken from his fig. 8), corresponding to our $\sigma = A^2 = 4$(SNR of +6 db). At the right side of fig. 23 we show where special points of Shannon's curves can be found in the utility chart.

When converting Shannon's magnitudes to ours one must consider that Shannon uses natural logarithmic units (nits) for the information rate. Thus his rate parameter R_{SH}* can be related to our bit density by the following derivation :

$$D = \frac{R}{B} = \frac{\log_2 M}{T_W} \cdot \frac{1}{B} = \frac{\log_2 M}{T_c B} = \frac{\log_2 M}{l\beta_c} = \frac{\log_2 e\,(\log_e M)}{\beta_c\, l}$$

$$= \frac{\log_2 e}{\beta_c} \,.\, R_{SH} \tag{75}$$

Using the minimum signal base per character (or dimension) of $\beta_c = 0.5$ (as Shannon assumes) we arrive at

$$D = 2(1.442695)R_{SH} = 2.885390 \,.\, R_{SH} \quad , \tag{76a}$$

or the reciprocal relationship

$$R_{SH} = 0.3465767\, D \tag{76b}$$

In decibels this relation can be written as

$$D^{db} = 4.602045 + R^{db}_{SH} \tag{76c}$$

With the help of these relations one can verify that Shannon's chart indicates a maximum rate of $R_{SH} = 0.8$ on the R_∞ scale for $E_L = 0$ (point P_1). This corresponds to a bit density of 2.3083 or +3.633 db. This indeed is the point in the utility chart where the $\sigma = 6$ db line cuts curve 1 (Shannon's upper bound, discussed in Part I of this paper).

To derive utility curves for nonbinary codebooks, one must use all the reliability curves of Shannon's 1959 paper. The following procedure was used to arrive at the curves of fig. 24 :

STEP 1 : Compute the reliability parameter from the given set of data : $\bar{e}_B$; l ; C(l, D) ; β_c . This can be done by solving equation 74 for E_L and by converting the word error ratio $\bar{e}_w$ to the bit-error ratio $\bar{e}_B$ in line with the frequently used upper bound $\bar{e}_B \leqslant \bar{e}_w/2$. The result is :

*To avoid any confusion with our transmission rate, R, we give Shannon's rate parameter the subscript SH standing for *Shannon*. Shannon's N_{SH} corresponds to l in this paper.

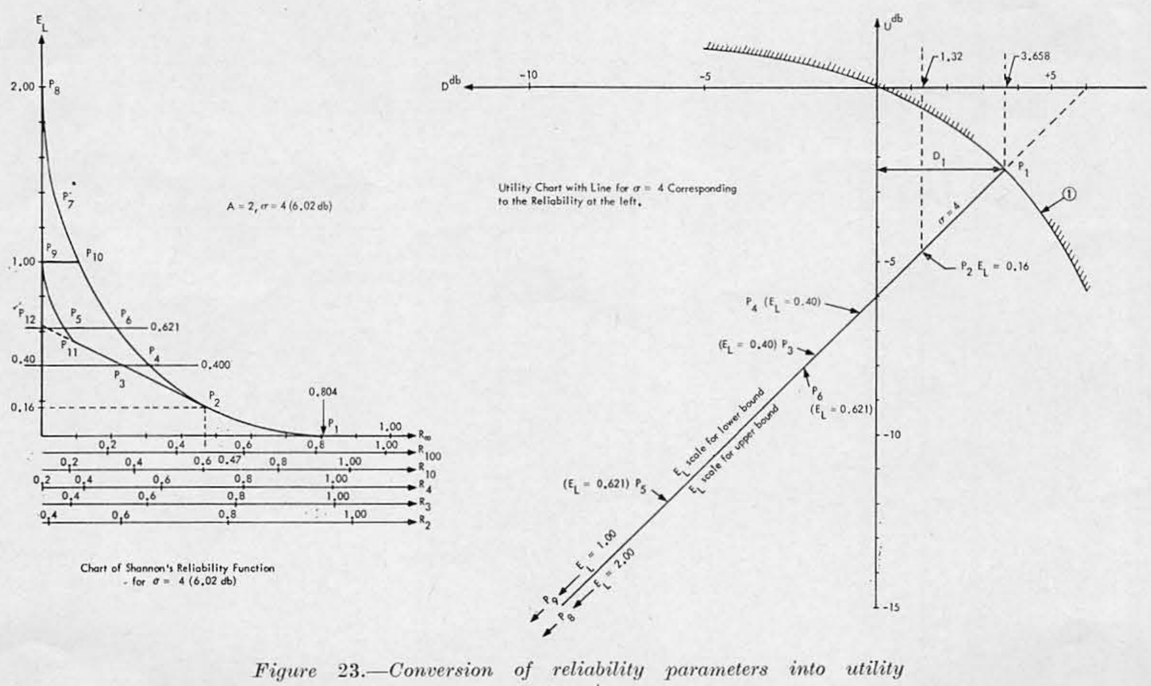


Figure 23.—*Conversion of reliability parameters into utility parameters.*

$$E_L = -\frac{\log_e(2\bar{e}_B) - \log_e C(l, D)}{l} \quad (77)$$

This is a constant value if $\bar{e}_B$ and l and C are kept constant. For the normalized presentations that follow we put $C(l, D) \equiv 1$ or $\log_e C(l, D) = 0$.

As an example we may assume $\bar{e}_B = 10^{-3}$ and $\log_e 2\bar{e}_B = -6.2146$. Assume that $l = 10$ makes $E_L = 0.62146$ for this case. Entering this value in fig. 23a leads to points P_5 and P_6.

STEP 2: Use the various charts of Shannon's paper and determine for each SNR ($\sigma = A^2$) the R_{SH} values for upper bound and lower exponential bounds. These values must be read from the correct R scale. For $l = 10$ one must use the R_{10} scale. The chart for $A = 2$ ($\sigma = 4$, $\sigma^{db} = 6.02$) yields for $E_L = 0.62146$ the R_{SH} values of $R_{SH} = 0.16$ and $R_{SH} = 0.37$. With the help of equation 76c (assuming $\beta_c = 0.5$) we find the corresponding values of $D = -3.36$ db and $D = +0.28$ db. The points P_5 and P_6 can now be found as the intersection of the ordinates with the $\sigma = 6.02$ db line. Any other β_c value would shift the D^{db} values for $3.01 + \beta_c{}^{db}$ to the left.

STEP 3: Continue to find the D values for all other σ lines for which $A^2 = \sigma$ is larger than $2E_L$. It can be easily seen that the highest E_L value on each of Shannon's charts is $A^2/2$. If any E_L value computed under step one is larger than $A^2/2$ for any given chart, the chart can not be used. In our example this is the case for $A = 1$ and all smaller A values. In terms of the utility chart this means that the utility curve for the upper bound of Shannon's model for $l = 10$, $\bar{e}_B = 10^{-3}$ is entirely to the right of the $\sigma = 0$ db line.

Instead of graphically finding the intersection of the vertical D lines with the diagonal σ lines, one may calculate the utility from the R_{SH} values and the A values of Shannon's charts by

$$u^{db} = R^{db}_{SH} + 4.602045 - 20 \log_{10} A \quad (78)$$

This process of converting Shannon's reliability charts to our utility charts has been completed for a number of l and $\bar{e}_B$ values. The results are shown in fig. 24. To avoid the overlapping of too many curves we omitted the above example $l = 10$, $\bar{e}_B = 10^{-3}$.

When discussing the meaning of fig. 24 we recognize that the curves for larger codebooks come closer to the utility of curve 1, Shannon's upper bound of the channel capacity. The difference between upper exponential bound and lower exponential bound (the shaded areas) is larger for small l values and for small $\bar{e}_B$ values. Both bounds become practically identical when the codebook is large and when a high error error-probability can be tolerated (e.g., the case $l = 100$, $\bar{e}_B = 10^{-1}$).

The practical importance of upper bound and lower bound is subject to various interpretations. One frequently mentioned interpretation says that no code of length l with nonbinary elements can be found that would give a better error ratio than $\bar{e}_B$ when the system operates with the bit densities (number of code words, M) and the utilities (reciprocal energy contrast) defined by the curves of the upper bound. This is the curve from P_1 to P_8, generally called the sphere-packing bound and originally derived by Elias (55). The same interpretation declares the lower bound as limit, indicating that at least one code must exist that makes the system operate with a better utility than the one indicated by the curve. This bound is actually computed from an upper bound on the average of all codes in the class, arguing that the best code in a class surely must be better than the average. This bound is known as the random coding bound and is also due to Elias (55) and Shannon (57). It coincides with the upper bound from point P_1 to P_2 in fig. 23a. Then it continues as a straight line to P_{11} and on to P_{12}.

From P_{11} to P_9 in fig. 23a there is a tighter lower bound called the expurgated random coding bound conceived by Shannon (57) and further exemplified by Gallager (65). The expurgated random coding bound reaches $R_\infty = 0$ at $E_L = A^2/4$. This point extends in fig. 24 into a σ line: $\sigma^{db} = E_L{}^{db} + 6.02$. These are all the solid diagonal lines in the left half of fig. 24.

When using this interpretation it is essential to remember that the curves are normalized for a coefficient, $C(l, D) = 1$. Small corrections will be needed in special cases where C deviates significantly from 1. Notice, however, that in equation 77, (l, D) may vary in a range from 0.1 to 100 without significantly influencing E_L when $\bar{e}_B$ is very small. Also, when l is very large, E_L becomes very small and its total influence on the utility curve becomes rather negligible.

Any point on the upper arm of the upper bound in fig. 23a (for example, the point P_7) requires a reliability that is higher than the highest reliability of the lower bound. This means that there is no assurance that a code can be found for such an operating point. It merely says that no code can be found with a still higher reliability. We therefore plotted this upper arm of the upper bound in dashed lines in fig. 24, cross-connecting upper bound and lower bound with a solid line, the so-called Plotkin bound (Plotkin 60). It is based on a bound on the average distance between

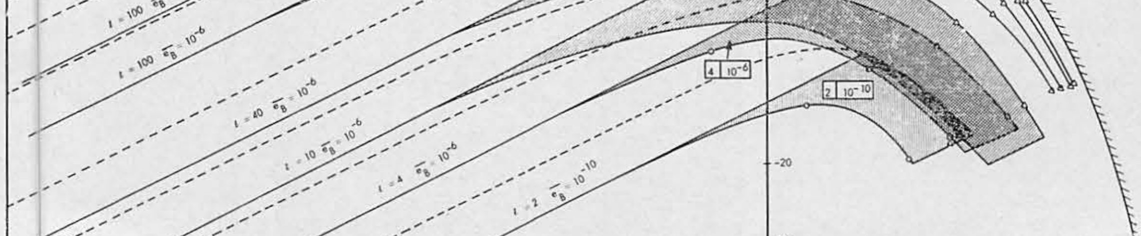


Figure 24.—Utility curves of systems with linear-real codebooks.

code words that can be found by adapting techniques originally applied to binary codes (Peterson and Kasami 65). The Plotkin bound is shown in fig. 23a as the straight line at $E_L = A^2/4$ between P_9 and P_{10}.

When charts from Shannon's 1959 paper are used, it is evident that they become inaccurate for extreme values of $E_L \to 0$ or $A^2 \to 0$. Fortunately, in the same paper Shannon derived two explicit equations that are accurate representations for the lower bound and the upper bound when σ is smaller than about -10 db. He also gave another equation that is valid as a joint upper and lower bound for rates close to the channel capacity. We shall now discuss these three approximations and convert them to the units of the utility chart:

Approximation 1: *Lower bound for* $\sigma < 0.1$

This is the random coding that is represented in fig. 23a by the straight line from P_2 to P_{11} and P_{12}. It can be seen from Shannon's reliability charts that the expurgated random coding bound degenerates into the normal random coding bound when the SNR $\sigma \ll 1$ ($\sigma < -10$ db). Points P_9, P_{11}, and P_{12} join in a single point. Shannon has plotted these bounds for small A ($\sqrt{\sigma}$ in our notation) in a special figure (fig. 13) in his 1959 paper. Peterson and Kasami (65) gave a larger version of the same curve. In these presentations the lower bound follows the equation

$$\frac{R_{SH}}{A^2} = 0.25 - \frac{E_L}{A^2} \tag{79a}$$

replacing $A^2 = \sigma$; $R_{SH} = 0.3466 \, . \, D$

$$\frac{0.3466\,(D)}{\sigma} = \frac{1}{4} - \frac{E_L}{\sigma};\quad \frac{D}{\sigma} = u$$

$$u = 2.885\left(0.25 - \frac{uE_L}{D}\right)$$

$$uD = 0.721348\,D - 2.885\,390 \, . \, u \, . \, E_L$$

$$u = 0.721348\,\frac{D}{D + 2.88539\,E_L} \tag{79b}$$

$$u^{db} = -\,1.41855\text{ db} + D^{db} - 10\log_{10}[D + 2.88539\,E_L] \tag{79c}$$

This is the equation for a utility curve when $\sigma = \dfrac{D}{u} < 0.1$.

It can be seen that this equation reaches maximum utility for $E_L \to 0$. The value is:

$$u^{db}_{\max} = -1.41855\text{ db} \tag{79d}$$

This value also occurred as the limiting value for binary codebooks in figs. 20 and 21. Indeed these figures are based on the same exponential bound.

Another interesting limiting case may be derived from equation 79 for cases where $D \ll 2.885\,E_L$, i.e., at the extreme left side of the utility chart. There we get

$$u^{db} \cong D^{db} - 6.0206\text{ db} - E^{db}_L \tag{79e}$$

Considering that $D^{db} - u^{db} = \sigma^{db}$, we can write equation 79e also in the form:

$$\sigma^{db} = E^{db}_L + 6.0206 \quad , \qquad \text{or } \sigma = 4E_L \tag{79f}$$

This is the limiting value for the diagonal σ line if $D \to 0$ ($D^{db} \to -\infty$).

Approximation 2: *Upper bound for* $\sigma < 0.1$

This is the sphere-packing bound. Shannon gives an approximate equation for the reliability of this bound under the assumption that $A = \sqrt{\sigma} \ll 1$. In Shannon's notations this equation takes the form:

$$E_L(R) \cong \frac{A^2}{2} - A\epsilon + \frac{\epsilon^2}{2}$$

The magnitude ϵ is related to Shannon's signalling rate R (again for $A \ll 1$) by the relation:

$$R \cong \frac{\epsilon^2}{2} \quad \text{or} \quad \epsilon \cong \sqrt{2R}$$

Eliminating ϵ from the approximate equation for E_L leads to

$$\frac{E_L}{A^2} \cong \frac{1}{2} - \sqrt{\frac{2R}{A^2}} + \frac{R}{A^2}$$

Shannon (59) in his fig. 13 and Peterson and Kasami (65) in their fig. 2 give fairly accurate normalized charts with plots of E_L/A^2 versus R/A^2. For many applications it may be desirable to solve this equation for the efficiency parameter of this paper, the utility u. This can be done with the help of equation 76, remembering that we added to Shannon's R (used in the equations above) the subscript SH to distinguish it from the parameter R used in this paper. The result of the conversion to the notations of this paper is the following approximate equation:

$$u = \frac{1}{2}\left[\frac{D}{(E_L - \alpha D)^2}\,(E_L + \alpha D - 2\sqrt{\alpha D E_L})\right] \tag{80}$$

The reliability function E_L, which can be determined from equation 77, will be a constant for fixed $\bar{e}_B$, C, and l. The constant α is known to any desired precision:

$$\alpha = \frac{\log_e 2}{2} = 0.34657359 \tag{80a}$$

Equation 80 may be expressed in the familiar decibel notation:

$$u^{db} = -\,3.0103 + D^{db} + 10\log_{10}[E_L + \alpha D - 2\sqrt{\alpha D E_L}] - \\ -\,10\log_{10}[(E_L - \alpha D)^2] \tag{80b}$$

The square on the inner bracket of the last term is retained to indicate that negative values of $E_L - \alpha D$ which frequently will occur do not disturb the formation of the logarithm. One could also write $20\log_{10}[\pm(E_L - \alpha D)]$ for this term.

Again one may further simplify equation 80 when assuming that $\alpha D \ll E_L$. This means one is interested in the utility curves of this model at the extreme left side of the chart where $D \to 0$ ($D^{db} \to -\infty$). In this range the utility is simply

$$u^{db} = -\,3.0102 + D^{db} + E^{db}_L \tag{81}$$

or

$$\sigma^{db} = 3.0102 + E^{db}_L \tag{81a}$$

The meaning of equation 81a is that the utility curves at the extreme left of the chart go into constant σ lines shifted for 3 db to the right against the line for $E_L{}^{db}$. Because αD must remain much smaller than E_L, one can see that this simple equation will be valid only for very small D. Under these circumstances this utility line corresponds to the point P_8 at $E_L = A^2/2$, except that the relationship of equation 81a would apply only to reliability charts for $A \ll 1$. The utility line of equation 81a corresponds also to the highest point, $E_L/A^2 = 0.5$, of the previously mentioned normalized charts of Shannon and Peterson.

Another simplification of equation 80b is possible when assuming $\alpha D \gg E_L$, provided we still operate in an area of the utility chart where $\sigma \ll 1$ (condition for the approximation in 80b). This means $\alpha D \ll 1$. Thus we can state that if

$$E_L \ll \alpha D \ll 1 \tag{81b}$$

we get

$$u = \frac{1}{2\alpha} = \frac{1}{\log_e 2} = \log_2 e = 1.4427$$

or

$$u^{db}_{\max} = +1.59206\text{ db} \tag{81c}$$

This is the highest utility that Shannon's 1948 bound for the channel capacity can ever reach (Part I) and that only for $D \to 0$. We can now understand the meaning of the restraint expressed by equation 81b. Yet it is gratifying to see that this explicit equation 80b for the upper bound of the utility of codebooks with nonbinary elements can reach the top utility. The question is therefore how small E_L must be made to come close to u_{max}; this, in turn, is related to the problem of finding the trade-off numbers between the dimensionality of the codebook and the loss of utility against u_{max}.

Approximation 3: Bound near channel capacity ($l \gg 1$)

Figure 23 shows that, between point P_2 and point P_1, the lower and upper bounds of the reliability (fig. 23a) and of the utility (fig. 23b) coincide. This characteristic of the reliability curves is valid for any A (or $\sqrt{\sigma}$) value. In the utility chart we may therefore expect to meet this combined bound along any constant σ line, no matter if σ is small or large. On the other hand, it is clear that this combined bound will always be the part of a reliability curve (or a utility curve) that comes closest to Shannon's curve number 1 representing the channel capacity.

Shannon (48), recognizing earlier work by Tuller (49) and Wiener (48), defines (in bits per second) the capacity of a continuous channel perturbed by white Gaussian noise as

$$C = W \log_2 (1 + \sigma)$$

In his 1959 paper Shannon uses the same channel capacity in nits (natural units of information) per degree freedom (dimension of the codebook) and he assumes that 2B dimensions per second may be transmitted. This gives (in nits per dimension) a channel capacity (designated by this author as C_{SH}) of

$$C_{SH} = 0.5 \log_e (1 + \sigma) \tag{82}$$

In most of our utility charts we plotted this channel capacity as a utility curve (usually curve number 1). It is defined by equation 21, repeated below, using the subscript "1" to identify these utility and bit density values as those that relate to Shannon's 1949 channel capacity:

$$u_1^{db} = D_1^{db} - 10 \log_{10} (2^{D_1} - 1) \quad ,$$

where D_1 is the expression $\log_2 (1 + \sigma)$ as may be seen from equation 18 in Part I. Therefore we may easily relate Shannon's 1959 notation C_{SH} to our D_1, using α from equation 80a:

$$C_{SH} = 0.5 \log_e 2 \,[\log_2 (1 + \sigma)] = \frac{\log_e 2}{2} D_1 = \alpha D_1 \tag{83}$$

Reliability charts are frequently expressed in Shannon's 1959 notation using *nits per degree of freedom* as the units of the channel capacity. Equation 83 gives the connection. The bit density D_1, corresponding to the channel capacity, may be read off curve 1 as the horizontal distance of any point on the curve from the vertical axis expressed in decibels. Converting this value to numbers and multiplying it with α gives the channel capacity C_{SH}. This has been done before in fig. 23b for point P_1. In that case,

$D_1^{db} = 10 \log_{10} [\log_2 (1 + \sigma)] = 3.658$ db or $D_1 = 2.3215$,

$\alpha D_1 = 0.804$ nits per dimension.

With this definition of C_{SH} and using R_{SH} (as before), the reliability near channel capacity may be expressed as follows (Shannon 59, equation 74):

$$E_L \cong \frac{(1 + \sigma)^2}{\sigma(2 + \sigma)} (C_{SH} - R_{SH})^2 \tag{84}$$

Introducing the equations for C_{SH} and R_{SH} yields:

$$E_L \cong \frac{(1 + \sigma)^2}{\sigma(2 + \sigma)} [\alpha^2 (D_1 - D)^2] \tag{84a}$$

Because $\sigma = \dfrac{D}{u}$, one may solve this equation for u, but the solution is a rather complex equation, since D_1 is also a function of u. Consequently, we prefer to express the bit density of the approximate combined bound as a fraction of the channel capacity D_1.

$$D = \delta D_1 ; \quad D^{db} = D_1^{db} + \delta^{db} \tag{85}$$

The factor δ is always smaller than "1", but close to "1". In decibels it will have small negative values. Introducing δ into equation 84a and using for $D_1 = \log_2 (1 + \sigma)$ leads to an equation for δ as a function of E_L and σ:

$$\delta = 1 - \frac{\sqrt{E_L \, \sigma (2 + \sigma)}}{\alpha(1 + \sigma) \log_2 (1 + \sigma)} \quad , \tag{86}$$

or in decibels:

$$\delta^{db} = 10 \log_{10} \left[1 - \frac{\sqrt{E_L \, \sigma (2 + \sigma)}}{\alpha(1 + \sigma) \log_2 (1 + \sigma)} \right] \tag{86a}$$

For any constant value of σ, one can express the equation in the simple form:

$$\delta^{db} = 10 \log_{10} [1 - \Delta(\sigma) \sqrt{E_L}\,] \quad , \tag{86b}$$

with the constant $\Delta(\sigma)$ defined by:

$$\Delta(\sigma) = \frac{\sqrt{\sigma (2 + \sigma)}}{\alpha(1 + \sigma) \log_2 (1 + \sigma)} ; \tag{86c}$$

and α from equation 80a has the numerical value:

$$\alpha = 0.34657$$

A few characteristic values are:

σ^{db}	-10	-6	-3	0	$+3$	$+6$	$+10$	$+20$
$\Delta(\sigma)$	8.653	5.294	3.642	2.500	1.980	1.216	0.831	0.500

With the help of these values it is easy to plot utility curves near channel capacity. E_L can be calculated from equation 77 for any fixed $\bar{e}_B$ and l. If a particular mathematical model is to be considered with C(l, D) significantly different from unity anywhere in the desired range for D, equation 77 still may be used, but E_L will now be a function of l *and* D. Once E_L is found, one draws, in a utility plot, the diagonal lines for various σ values; for example, for those listed in the table above. Using the precalculated $\Delta(\sigma)$ values, or calculating them for other σ values with the help of equation 86c, and applying 86b gives δ in decibels. This value is the distance on the horizontal axis that one has to go to the left from D_1 (the intersection of the σ line with the curve 1) to find the D value of the desired utility curve for the particular σ value. The corresponding u value results as the intersection of that vertical D line with the σ line or from the relationship $u^{db} = D^{db} - \sigma^{db}$. Thus one can plot a utility curve point by point by assuming reasonable σ values and calculating the corresponding D and u values.

Figure 25 shows a number of interesting utility curves computed with the help of these three approximate formulas. The curves must be computed for large l values to keep them at the left side below -10 db signal contrast and to keep them on the right side close to channel capacity. These curves demonstrate the advantages of nonbinary codebooks with sufficiently long code words. The number of code words that are required to meet a given bit density can be calculated from the definition of bit density as

$$D = \frac{R}{B} = \frac{n}{T_w}\left(\frac{1}{B}\right) = \frac{\log_2 M}{T_c \, l B} = \frac{\log_2 M}{\beta_c \, l} ; \tag{87}$$

$$M = 2^{\beta_c l D} = 2^n \quad . \text{ (See also equation 65.)} \tag{87a}$$

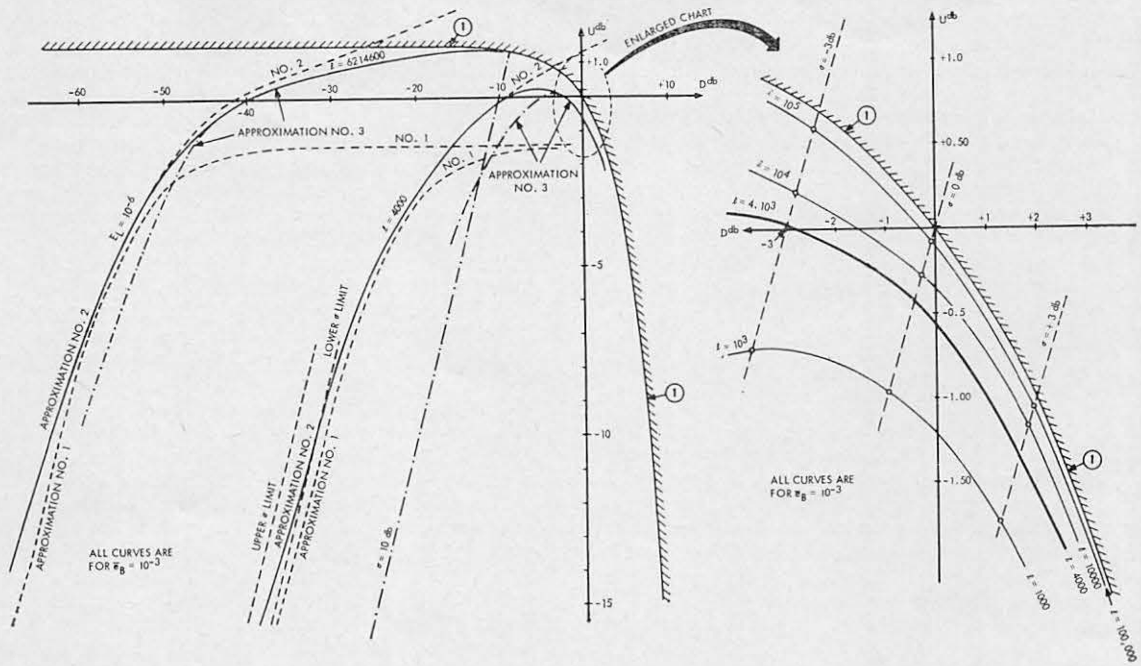


Figure 25.—Utility curves defined by the approximate equations for an ideal system with linear-real codebooks.

Figure 25b shows the interesting section $-3 < D^{db} < +3$ in a larger scale. In this range the curves follow the third approximation.

Apparently M will be unrealistically large for any attempt to arrive at the right side of the utility chart close to the channel capacity. The practical engineer is therefore most interested in utility curves with small l, so that he can keep the whole codebook in reasonable proportions. Unfortunately, none of the approximate formulas are valid for small l. The accurate expressions for the error probability all depend on the error function (probability density integral) and thus can not be expressed in the explicit form $u = f(D)$. Several authors, however, performed computer calculations and arrived at charts that can be used for graphical evaluation, similarly as we used Shannon's charts.

In Part I, fig. 5, we used Slepian's curves as an example for the utility curves of ideal nonbinary systems. These curves have been converted to our notation from Slepian (63). In fig. 5 only the upper bound is plotted, but this is done for two different error ratios. Now we would like to show in fig. 26 the wide spread between upper bound and lower bound when l has a small value. To avoid overcrowding the chart, we do so only for one fixed error ratio ($\bar{e}_B = 5.10^{-5}$).

Slepian (63) has included the reliability function and the coefficient of the exponential bounds in all his computations. He plots curves for a fixed error ratio exactly as we do. When converting his word error ratio to the bit error ratio of this paper, we again use the relation $\bar{e}_w = 2\bar{e}_B$. Since Slepian publishes his curves only for a few fixed word error ratios, we used $\bar{e}_B = 5.10^{-5}$ as the fixed value for our fig. 26, corresponding to his $P_e = 10^{-4}$. In fig. 5 in Part I we applied a graphical correction to arrive at bit error ratios of 10^{-4} and 10^{-6}.

The vertical axis of Slepian's curves is the SNR difference

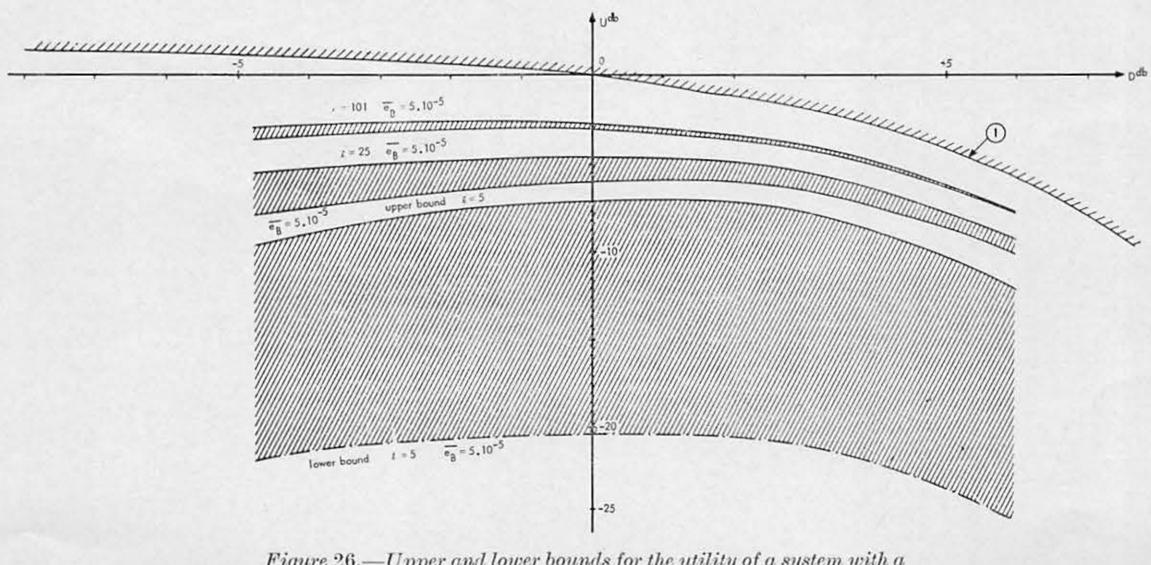


Figure 26.—Upper and lower bounds for the utility of a system with a fixed number of dimensions (after Slepian).

of an ideal system with a nonbinary codebook of n-dimensions (l in our notation) against Shannon's ideal system with an infinite number of dimensions (our curve 1). The units of the vertical axis are in decibels and we may call this σ difference η.

The horizontal axis in Slepian's curves (r) is the bit density as defined in this paper, but the scale is based on the common logarithm in place of the logarithm to the base " 2 " as used herein. His units appear therefore in dits per Hz bandwidth. This forces the engineers alternately to use bits, nits, or dits, depending on the individual author. (We hope the unified presentation in this paper will be appreciated.)

The conversion formulas from Slepian's notations to ours are (SL standing for Slepian):

$$u^{db} = u_1^{db} - \eta_{SL}^{db} = D^{db} - \eta_{SL}^{db} - 10 \log_{10} (2^D - 1) \qquad (88)$$

$$D^{db} = 5.2139 + 10 \log_{10} r_{SL} \qquad (89)$$

Figure 26 shows the wide spread between upper and lower bound at low l values and for bit densities around zero decibels. Slepian, however, shows in his fig. 2 that many known codes give results closer to the upper bound than the lower bound. This is to be expected as the lower bound represents the average performance of all possible arbitrarily selected codes of length l, including the worst ones. The difficult problem is apparently to derive an algorithm for finding at least one good code and at best for identifying the optimum code for any given length l. Slepian (68) addresses exactly that problem and, in his concluding remarks, he points out that his paper may have " raised more questions than [it has] answered ". He further concludes that " there is a great abundance of groups of arbitrarily large order than can be examined from the point of generating group codes " and he develops a number of theorems that will be of great help to research workers in solving the important problem of finding optimum nonbinary codebooks.

A closely related contribution to the graphical presentations of the mathematical models of systems with nonbinary codebooks comes from WJ (65). We already used their exponential bound parameter R_o in equation 89 for the discussion of binary codebooks. For the treatment of nonbinary codebooks, WJ introduce on page 311, equation 5.44b, a nonbinary exponential bound parameter that can be related to the units of this paper.

The results of such effort are given in fig. 27 for one fixed bit error ratio ($\bar{e}_B = 10^{-3}$). It can be seen that the utility curves gained from WJ's charts supplement well the curves gained from Shannon's reliability function and from Slepian's charts. A numerical comparison shows that the WJ charts are slightly below Shannon's upper bound but well in line with Slepian's bounds. They cover a much wider range of bit densities and code lengths than either of the other sources. This seems to bring all three sources in very good agreement. Notice that WJ derive their exponential bound as a lower bound and that Slepian proved that many known codes are considerably above his lower bound. It may therefore be fairly safe to use the WJ curves as a guideline for the utility of systems with nonbinary codebooks when operating with ideal circuitry over a Gaussian channel.

All these results have been anticipated by theoretical investigations and some of these results are contained individually in any one of the many contributions of other authors. It is felt, however, that the utility chart offers to the practical engineer the advantage of seeing the potential tradeoff gains and costs all in one plot.

Naturally, the engineer will also be interested in obtaining the tradeoff between the error ratio and the other design parameters. For this purpose we give in fig. 28 the result of using the WJ charts when plotting curves for codebooks with constant length, but operating such an ideal system with different transmission quality. For facilitating the comparison with earlier (binary) charts we chose a code length of 40 dimensions and plotted charts for seven different bit error ratios ranging from 10^{-1} to 10^{-10}. The cost in utility for decreasing the error ratio over this whole range is nearly 14 db at a bit density of -15 db. At the maximum of the curves (around zero db bit density) the spread is only 6 db and it decreases further at the right side of the chart. A comparison with the curves for a very long codebook ($l = 1000$) shows that large codebooks not only offer the advantage of high utility but they also assure the highest quality of information transmission without any significant penalty in utility. The two curves for $l = 1000$, $\bar{e}_B = 10^{-1}$, and $l = 1000$, $\bar{e}_B = 10^{-10}$ coincide over most of the right side of the utility chart.

Also interesting is the comparison of the curves of fig. 28 with the utility lines of an ideal binary system (equation 34 in Part II). For this purpose we inserted dashed lines

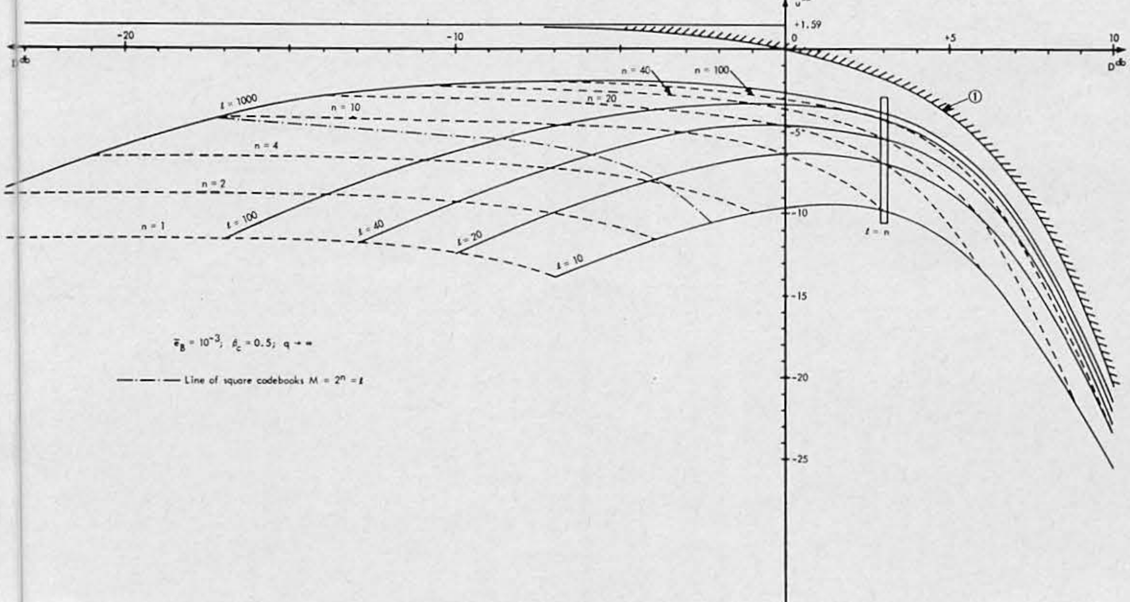


Figure 27.—Lower bound for the utility of a system with a fixed number of dimensions (after Wozencraft and Jacobs).

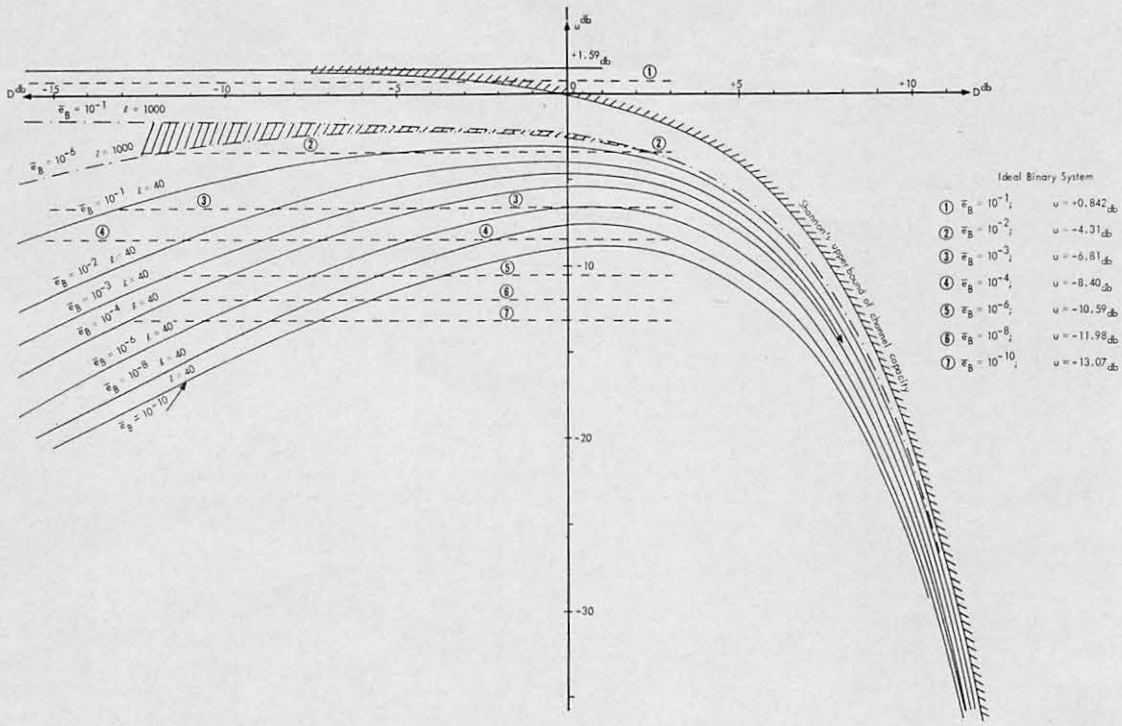


Figure 28.—Utility curves of ideal multidimensional systems for various error ratios.

for the utility of an ideal one-dimensional binary system for the same error ratios as the ideal multidimensional nonbinary system. One can see that for high error ratios (low transmission quality) the complex multidimensional nonbinary system does not offer any significant advantage.

After having discussed the most general mathematical models of multidimensional nonbinary systems, it is important to show what kind of codes could be used to implement such systems. After Rice (50) and Shannon (48, 49) had originally suggested using codes with randomly selected real numbers as code elements, Gilbert (52) was possibly the first to investigate some simple nonbinary codebooks with integer elements. He explains some ternary alphabets, called Slepian alphabets, that use 2 M dimensions (code length) and words with M elements having letter O and M elements having either $+1$ or -1 as letters. A typical example for $l = 10$ is a code word $(+\ 1\ 0 + 1 - 1\ 0 + 1\ 0\ 0\ 0)$. Gilbert shows that such nonbinary codebooks can be very efficient. Systematic coding procedures for digit decoding using nonbinary code elements have been suggested by Golay (49a), and a few fundamentals for nonbinary shift register codes have been included by Golomb (55) in his basic report on digital sequences. Loomba (61) published an interesting comparison of nonbinary BC codes in digit decoding versus binary orthogonal codes in message decoding. These pioneering efforts and the many efforts on nonbinary error-correcting codes that followed them are not of direct concern in this section. All the models that have been discussed above are based on block codes with MLD decoding procedure. This requires message decoding in contrast to the usual symbol decoding. Yet certain nonbinary error-correcting codes could be used as nonbinary codebooks in our sense. The review of such possibilities is beyond the scope of the present paper.

There are, however, two approaches to more restrictive mathematical models that have been successfully brought into the form of numerical charts. The first approach deals with nonbinary codebooks that use multilevel elements. It is due to Wozencraft and Jacobs and is published in their book in the form of two charts (figs. 5.17 and 5.18), giving the exponential bound parameter as a function of the energy ratio per dimension. We have extended WJ's fig. 5.17 to a wider range of E_N/N_o, called x in our fig. 29. This variable x is related to the variables of the utility chart by the equations shown in fig. 29. The exponential bound parameter R_o^*, as a function of x, is plotted in fig. 29 for various values of q, the number of amplitude levels that may be used for the elements of the nonbinary codebook. Evidently the curve for $q \rightarrow \infty$ is identical with the curve used for plotting figs. 27 and 28.

There are two useful approximations to the curve R_o^* (x, $q \rightarrow \infty$). They are given in fig. 29 by the equations 2 and 3 while equation 1 is the analytical form of the correct curve for $q \rightarrow \infty$. With the help of this extended chart drawn to a very accurate scale, we plotted in fig. 30 the utility charts of quantized nonbinary codebooks, used as multilevel codebooks. The utility curves of fig. 30 are plotted for $\bar{e}_B = 10^{-3}$ and $l = 40$. The signal base is $\beta_c = 0.5$, as usual. The curve for $q = 2$ is identical with the curve for the same parameters plotted in fig. 22 for binary codebooks. The same exponential bound has been used in both cases. The curves for $q > 2$ show that the principal advantage of multilevel nonbinary codebooks is in an extension of the utility curves to larger positive bit densities. The differences in the utility of the various curves at the left side are due to some geometrical characteristics of quantized real codes in the l-dimensional vector space. WJ developed an optimization procedure for assigning different probabilities to the different amplitude levels when selecting them for elements in the code words. With the help of this procedure all curves may be brought to the same level at the left side, i.e., to the level of the curve for $q \rightarrow \infty$. Figure 5.18 in WJ shows the R_o^* resulting in that case.

Figure 30 shows Shannon's upper bound for this case, gained from Slepian's curves by interpolation. Remember that the WJ model is a lower bound. To show the wide margin that still prevails between the lower bound of the theoretical models and the practical results of one of the first experimental systems that have been tested, we also inserted the operating line for the Lincoln Laboratory's

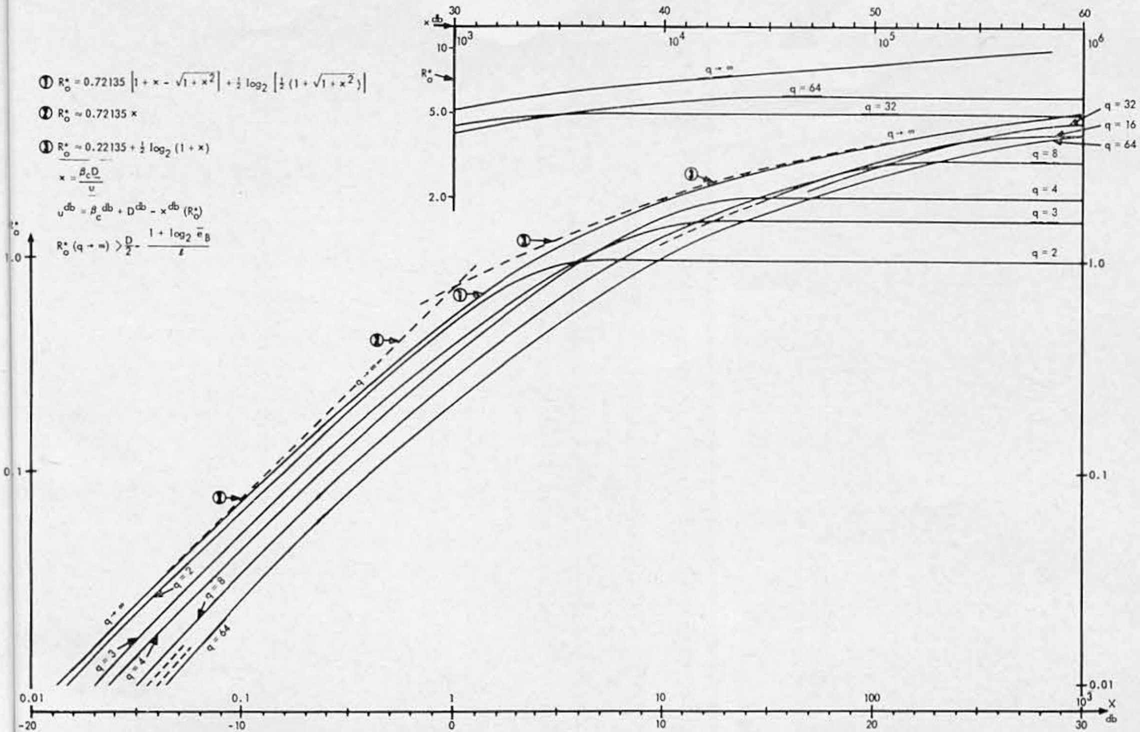


Figure 29.—Plot of the reliability of multidimensional systems with quantized elements.

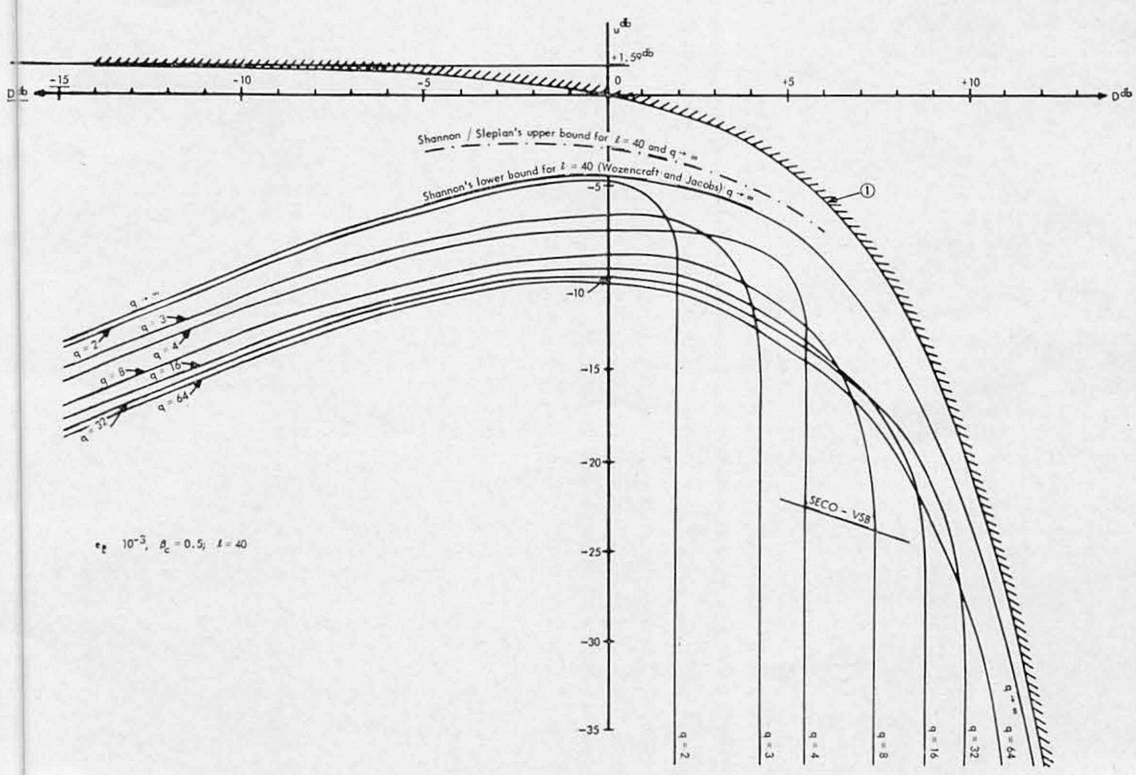


Figure 30.—Utility curves of multidimensional systems with quantized elements.

self-regulating error correction coder-decoder (SECO) system that operates with q = 32 levels (section F7, Part II). It had been classified as a one-dimensional system because it uses symbol decoding and not message decoding. For its class the SECO is efficient equipment, but the combined inefficiencies of the digit decoding method, the waveform spectra, the synchronization methods, and many other engineering problems place it in utility about 10 db below the mathematical model. Interesting, however, is the fact that a system of this complexity has been built at all and that in its first attempt it comes that close to the theoretical optimum. Notice also that the slope of the operating line of the SECO system is smaller than the slope of the mathematical model (the utility chart marked q = 32). This makes the practical system more efficient at higher bit densities than at lower ones. There are many possible reasons for this behaviour; some of them could be in the kind of measurement methods applied for the tests of SECO.

The second approach to arrive at numerical charts for more restrictive mathematical models of systems with nonbinary codebooks is due to Peterson and Kasami (65). These authors compared the following classes of nonbinary codebooks:

A. S-Classes: in line with Shannon's (59) definitions.

B. D-Classes: generally called polyphase codes and also investigated by the following authors: Viterbi (65), Stiffler and Viterbi (65), and Reed and Scholtz (65, 66).

C. C-Classes: These are double encoded systems where the β-elements of the first codebook (fig. 16) are further encoded into a waveform library (not necessarily of a finite alphabet) where each waveform consists of an M-tuple of real number components such that all waveforms have the same average power, and thus are represented by vectors with end points on an M-dimensional sphere.

Peterson and Kasami (65) use exactly the same reliability charts as Shannon (59) and as we discussed in fig. 23. Their results can therefore directly be converted to utility charts following the previously described procedure. The results indicate that, for very small SNR (power contrast σ), all the classes above merge into the same curves for lower and upper bound or for the combined bound. These bounds are the ones that we derived as approximation for $\sigma \ll 1$ from Shannon's (59) paper and that we reproduced in Fig. 25. For σ values closer to unity and above unity, we find that again all curves merge for the same bound into the same curve or point when the bit density goes to zero, i.e., at the left side of the curve.

The principal results of Peterson and Kasami (65) are therefore the differences in the reliability E plotted in the charts for $A = 1$ ($\sigma = 0$ db), $A = 2$($\sigma = +6.02$ db), $A = 4$($\sigma = +12.04$ db), $A = 8$($\sigma = +18.06$ db), and $A = 16$($\sigma = +24.08$ db) for values of the rate parameter larger than 0.1 ($D > -5.4$ db). This area corresponds to the centre and the right side of the utility chart, again an indication that nonbinary codebooks are of importance for systems with high bit density. In that area we recognize that S-classes (i.e., the real linear codes), have the highest reliability (and, correspondingly, the highest utility for any given signal contrast σ) of all the classes, no matter whether one takes the lower bound or the upper bound. The C-classes (i.e., the double encoded systems) are significantly better than the D-classes and in both classes the size of the waveform alphabet (q) becomes important at the higher bit densities. Naturally, the tendency is toward increasing utility for increasing q value. Beyond these general observations the reader should have no difficulties in finding, for any particular selection of a class of codes and for any specific set of values for $\bar{e}_B$, l, q, and D, the utility u by using equation 76 (Peterson and Kasami use the same R as R_{SH} in this paper) and equation 77 and the corresponding curve (for specific q value and for the code class) in Peterson.

There are many other important investigations of the mathematical background of nonbinary codebooks. The previously mentioned paper by Gallager (65) gives a most complete coverage of the whole field. Much interest resulted from the so-called n-orthogonal codes that were investigated by Viterbi (65) as polyphase codes, by Stiffler and Viterbi (65) as q-orthogonal signals, and by Reed and Scholtz (65, 66) as n-orthogonal phase modulated codes. Multiphase codes for radar purposes were proposed earlier by Heimiller (61) and Frank and Zadoff (62). N-orthogonal codes are closely related to the simplex codes (to be discussed in the next section) as pointed out by Weber (66) and I. Jacobs (67). The latter publication offers a very interesting comparison of n-orthogonal phase modulation with many other nonbinary systems and concludes on page 83 " that n-orthogonal phase modulation is generally inferior to more conventional techniques ".

Interesting progress towards specific nonbinary codebooks may also come from a further expansion of the theory of binary sequences to sequences with nonbinary elements. Kurz (61) postulated an efficient code, the *Minimax code*, that may form the basis for future developments of nonbinary codebooks combined with orthogonal waveform libraries, possibly along the lines of Peterson and Kasami (65) class C of codes. Hsieh and Hsiao (64) show a method of constructing n-ary codes, particularly ternary codes from binary sequences using orthogonal functions. Much hope may also be placed on the *generalized Barker Sequences* which received a thorough fundamental treatment by Golomb and Scholtz (65). Barker sequences were originally suggested for group synchronization of binary digital systems (Barker 53). Delong, Jr. (59) extended their theory to three-phase codes. The generalization by Golomb et al introduces complex numbers having an absolute value of " 1 ". This makes them very suitable for codebooks of the D-class of Peterson and Kasami (65). Of additional interest may be a note by S. H. Chang (66b) on nonbinary orthogonal codes and the work by Levitt and Wolf (65) on the correlation properties of multilevel cyclic sequences.

The theoretical superiority of codes that permit any real number, not just integers, to function as code elements β_l has been established. Our fig. 30 shows the advantage to be gained by going with q toward infinity. The practical engineering possibilities to implement such codebooks are under investigation by W. H. Pierce (68) for the linear-real codes (originally anticipated by Kelly (60) and then devised by W. H. Pierce (66). Schneider (68) investigated the engineering possibilities for Slepian's (65a) permutation modulation. Ziv (63) investigated the coding and decoding for time-discrete amplitude-continuous memoryless channels in a doctoral thesis and his results should be of interest to developers of linear-real coding equipment.

Concluding, we may predict that the multidimensional information transmission systems with nonbinary codebooks will become of growing importance in the future. Their mathematical models are well developed. The code generators will have to operate on a systematic basis to avoid excessively large memory requirements for codebooks. The codes will require code lengths of one hundred and more elements and the elements or samples will have to be selected from large alphabets.

K8. *Mathematical Models of Systems with Simplex Codebooks*

In subsection J2 we gave a capsule history of the sphere-packing problem and mentioned that regular polytopes (Coxeter 48, 62) were investigated as the geometrical figures in an n-dimensional vector space that would lead to " good " codebooks in the sense of having the maximum average distance between all code words. First Gilbert (52), and

later Basore (59) and Stutt (59, 60), developed regular polytope codes. The term " polytope " should be preferred to the term " polyhedron ", which is reserved for a polytope in three-dimensional space. The term " simplex " designates a subclass of n-dimensional polytopes with elements that are simplexes of n-1 dimensions. A regular simplex is basically a simplex with all " edges " being equal. The exact definitions of these terms in n-dimensional space are more complex and should be checked in the book of Coxeter (62).

Stutt (59) gave a procedure for developing an " equilateral polyhedron " (regular simplex) in a minimum number of dimensions and computed a number of charts that can be easily converted to utility charts. Regular simplex codes have utility curves which are only slightly above the utility curves for orthogonal codes (figs. 20 and 21). Their codebooks are nearly square with $l + 1$ words for l dimensions. This gives them a slightly larger bit density for any given l than orthogonal codes can offer. Simplex codes have a slightly negative cross-correlation matrix and therefore give a slightly higher utility than orthogonal codes when operated with an MLD processor.

Balakrishnan (60) showed that (a) when one out of a finite number of signals (represented by vectors in l-dimensional space) is transmitted over a Gaussian channel, the optimum l-dimensional complex for placing the signal vectors is the regular simplex, and that (b) all signal vectors should end at the corners of the regular simplex. Balakrishnan (61) investigated the influence of the SNR (defined in geometrical terms) on the optimal choice of a codebook and proved that the regular simplex is a *local maximum* independent of the SNR. Balakrishnan and Taber (62) gave error probability curves and curves for the communications efficiency of simplex systems and compared the results for simplex systems with those for biorthogonal systems, showing again that the regular simplex coding yielded the lowest possible error ratio for a given SNR (defined as signal power over noise power density). Balakrishnan (65) related his earlier results to the problem of signal selection for space communications channels and stressed again the need for proving that the regular simplex is an " absolute or global " maximum independent of SNR, or for disproving this conjecture.

Landau and Slepian (66) proved finally that the regular simplex code is indeed the optimum code for all SNR when the codebook matrix has l dimensions and $M = l + 1$ code words. They also showed that the biorthogonal code likewise is optimum when the code book has six words and three dimensions ($M = 2l$) and that a code consisting of the midpoints of the faces of the regular dodecahedron is also optimum for $M = 12$ and $l = 3$ ($M = 4l$). " Optimum " in all these derivations means that the application of these codes over a Gaussian channel in any system with an MLD receiver and otherwise ideal circuitry will yield the lowest possible error probability for any given SNR.

L. Experimental Multidimensional Systems

Compared with the strong development effort that takes place in the area of one-dimensional nonbinary systems, there is not yet much activity in experimental multidimensional systems.

L1. *DIGILOCK Data Transmission System*

The development of a system using binary orthogonal sequences as nonbinary signals was widely publicized. Called the DIGILOCK system, it is based on a mathematical model published by Viterbi (61), which was used for the utility charts of fig. 2, Part I. The development work on DIGILOCK was originally sponsored by the Jet Propulsion Laboratory, and the fundamentals of the system were published by Sanders (59a ; 60b). The experimental version of the system was designed to operate with orthogonal sets of a modified Reed-Müller code type (first order) or with the same set in biorthogonal version. An MLD receiver was used. Viterbi (61) showed how codes derived from PN sequences can be used as orthogonal sets. Sanders (65) gave another review of the DIGILOCK fundamentals and discussed the synchronization problem and its interconnection with the choice of the waveform alphabet (binary, in this case). From all these reports we learn that biphase modulation is used on the radio carrier in the experimental version. It is reported that the laboratory test results placed the utility 2 db below the theoretical value (fig. 2, Part I). No information is supplied about the bandwidth occupancy ; thus it is impossible to define the bit density exactly. The values used in fig. 2 seem to be a fair assumption for biphase modulation. It is not reported that the experimental version of DIGILOCK had been tried in field tests with synchronization over a noisy channel. It is, however, reported that a special version of the DIGILOCK telemetry system passed its flight test on December 4, 1961 (Collins 61). A more complete description of this operational system (Jaffe 62) revealed that the system was used on the Blue Scout, Jr. USAF rocket in parallel with a standard FM/FM telemetry link. A 16-dimensional modified Reed-Müller code was used in the biorthogonal mode. Phase modulation of $\pm 60°$ was used on the radio frequency carrier to keep sufficient carrier power at the carrier frequency itself, so that a phase lock loop could operate safely at the receiver.

During the field test the system operated with a total average transmitter power of 250 mW. The maximum range is 100,000 miles ; the transmitter antenna gain, 0 db ; and the receiving antenna aperture area, 850 ft^2 = 79 m^2. This gives a signal power at the receiver input (Filipowsky and Muehldorf 65a) of

$$P_r = \frac{P_T A_r}{4\pi R^2} = \frac{0.25\,(79)}{4\pi(2.59)\,(10^{16})} = 5.96 \times 10^{-17}\,\mathrm{W} \qquad (90)$$

The noise power at the receiver input may be calculated from the receiver bandwidth B and from the receiver noise figure NF increased (after conversion) by antenna and coupling device noise temperature, using the formulas :

$$P_N = KT_N B\,; \qquad T_N = 290(NF - 1) + T_C \quad , \qquad (91)$$

with $K = 1.38(10^{-23})$ Watt second per degree Kelvin.

Jaffe gives a noise figure of 2.82 and a bandwidth of 3200 Hz . We may assume a noise temperature of about 70° for the external noise contributions, making the operational noise temperature about 600°K.

$$P_N = 1.38(10^{-23})\,(600)\,(3200) = 2.65(10^{-17})$$

This allows a power contrast of $5.96/2.65 = 2.25$ or $\sigma = +3.52$ db. Before we can calculate the utility, we need the bit density. The system operates with 67 information bits per second, but every 21st word is a frame synchronization word making the actual message rate $R_m = 67(20/21) = 63.8$ bps. The bit density is therefore

$$D = \frac{R_m}{B} = \frac{63.8}{3200} = 0.0199 \quad \text{or} \quad D^{db} = -17.01\,\text{db}$$

Notice that the system uses a codebook with $l = 16$ dimensions and $M = 32$ code words. The signal base per dimension is rather large :

$$\beta_c = BT_c = B\frac{T_W}{l} = \frac{B\log_2 M}{l\,R} = \frac{3200(5)}{16(67)} = 14.9$$

Notice that, for the calculation of the signal base, we must take the bit rate of 67 bps (also called the throughput) ; whereas, for calculating the bit density D, we can only count the transmitted message bits, deducting any bits for frame synchronization ; these bits contribute to the total synchronization and acquisition performance. If these extra bits are for some " housekeeping " purposes and can be freely used for messages without disturbing the performance of the system, the message rate should be equal to the throughput.

Knowing D and σ for the operation of the system at the maximum range, we can now find the utility as

$$u^{db} = D^{db} - \sigma^{db} = -17.01 - 3.52 = -20.52 \text{ db}$$

The system uses roughly 25 per cent of the transmitter power for the carrier channel (phase lock loop bandwidth = 20 Hz). This requires that 1.9 db be deducted from the operational utility. The operating point of the system, when operating with full power, is therefore at u = −20.7 db and D = −17 db. Because the designers need a design margin, they specify 6 db. Thus the utility should not go below u = −14.7 db. At this place the system still has an operational margin of about 6 db against the design goal of u = −9 db, where the theoretical error ratio should be 10^{-6} or better. That design goal was specified as $\beta_{SA} = 8$, using Sander's communications efficiency (Sanders 59d, 60a) which is (with reservations) the reciprocal value of the utility.

The actual receiving technique used a predetection recording method and a signal processing maximum likelihood decoding procedure with the help of a commercial computer. By these rather unconventional methods it was possible to make full use of the large signal base (by integrate and dump procedures) and to perform truly an MLD decoding operation. It was interesting for the designers to test the receiver portion under worst case conditions by reducing the SNR for a laboratory test. They adjusted the SNR in a 3200 Hz band after a crystal filter to the low value of −12.0 db, corresponding to −13.2 db for the message channel above. They achieved an error rate of 0.02 per word or 0.01 per bit (now counting the frame synchronization bits as message bits). The decoding was performed by the computer. This worst case situation yields a utility of u = −16.8 + 13.2 = −3.6 db. This is only about 2.2 db below the theoretical value of u = −1.4 db, which could be achieved with D = −16.8 db and $\bar{e}_B = 10^{-2}$.

L2. *Block Coded Telemetry Systems of Jet Propulsion Laboratory*

The staff of the Jet Propulsion Laboratory has consistently improved their experience with multidimensional nonbinary systems. Much of this experience resulted from original work on ranging codes (59, 60, 65). A system block diagram and some logical design information about a biorthogonal telemetry system of 16-dimensions for 45 bps are supplied in another report (61a). To improve the demodulator of the MARINER 2 data system, JPL made an investigation to find practical design approaches to matched filter correlators (Futz 63). Futz's (63) note concludes that the usual mechanization of a coherent demodulator is not equivalent to the commonly used mathematical model. The investigation showed that it is possible, by correctly matching a filter to the signal waveform, to improve the performance of the Mariner 2 demodulation by 0.9 db. The practical realization of the matched filters, however, presented difficulties.

An interesting article (63b) describes a large number of trial runs of a biorthogonal telemetry system (published earlier in JPL publications). The results of these trial runs performed in the laboratory with a noise generator showed that the SNR ratio to obtain a particular error probability was within 0.3 db of that predicted theoretically. The system subjected to these trial runs was the 45 bps telemetry system mentioned above. It was exactly timed to a word period of 110 ms, to a bit period 22 ms (n = 5), and a chip period of 6.875 ms (l = 16). The chips biphase-modulated a 4 KHz rectangular carrier. A receiver band-filter restricted the transmission bandwidth to 5.4 KHz (1.3 to 6.7 KHz). This gives a bit density of

$$D = \frac{1}{T_B B} = \frac{10^3}{22(5.4)\,(10^3)} = 0.00841 \text{ (or } -20.74 \text{ db)}$$

The utility (or its reciprocal value E/N_o) is shown as a function of the bit error probability, and the experimental values agree within a fraction of a decibel with the theoretical values. The results of these laboratory tests are gratifying since they confirm the correctness of the mathematical model. Yet, the same problem arises as in the case of the tests of the DIGILOCK system reported by Jaffe (62). The bit density in the tests actually used is −20.74 db. The reasonable minimum bit density for a biorthogonal system of n = 5 can be taken from fig. 2 as about −4 db. (The theoretical minimum bit density for n = 4 in an orthogonal system would be −3 db in a low-pass channel with sin x/x waveforms, corresponding to $\beta_c = 0.5$). Because of the phase modulation the designers preferred the much larger $\beta_c = n/lD = 37.2$ (or 15.7 db), but, since they use it only as a correlation improvement (like in ideal binary systems), no improvement in utility results from this large bandspreading effort. The note gives no information about field tests or tests on space flights but it discloses details on the development of the digital circuit modules. It may be of interest that the decoder, similarly as in the USAF DIGILOCK system, is a digital signal processor which resamples each received chip 55 times and quantizes each sample into 11bit words (2048 levels). This comes very close to an ideal MLD receiver. It is therefore no suprise that the laboratory tests are indeed within 0.3 db of the theoretical model.

Springett (65) stressed the importance of the JPL research in multidimensional nonbinary systems and in related areas for the further development of practical space telemetry and command systems. It is interesting to note that he uses a normalized SNR defined as the signal power over the noise power density S/N_o, a magnitude defined in equation 12 in Part I as the characteristic rate of a system and designated by the symbol "ρ". The importance of the parameter has been stressed independently of Springett by Filipowsky and Muehldorf (65a), where it was called signal contrast frequency. Contrary to the truly normalized SNR terms, the power contrast (σ) and the energy contrast ($\epsilon = E/N_o$), this so-called "normalized SNR" is actually a physical magnitude with the dimension of a frequency or a rate, i.e., with the dimension (second)$^{-1}$. We therefore suggest calling it "*characteristic rate*", not "normalized SNR". This term would also be significant since it can be easily shown (in addition to the relations in equation 12) that this magnitude ρ is related to the utility in the following form :

$$u = \frac{D}{\sigma} = \frac{R}{B} \cdot \frac{N}{S} = \frac{R}{\rho} \tag{92}$$

Springett (65) shows that the state-of-the-art at that time permitted communications from MARS to Earth at ρ = 30.5 db-Hz, or at a rate of 1121 bps, with a hypothetical system that could operate with a utility of unity (0 db or 100-percent efficiency). The numerous references in Springett (65) will be helpful to readers interested in the engineering evolution of space data transmission systems.

Saliga (67) reports an advanced space telemetry system with biorthogonal codes of 127 dimensions carrying 8-bit words. The experimental system was developed along the lines of DIGILOCK and of the JPL telemetry systems for 5-bit words that we discussed above. The significant improvement is due to a new synchronization and timing subsystem, the *Carrier Coherent Telemetry Rate* (*CCTR*) subsystem. The improved synchronization, acquisition, and timing methods are based on the experience with ranging codes (Titsworth 64). These methods are explained in detail in Saliga's 1967 paper. The result of these improvements is an acquisition threshold at u = −0.5 db and a worst case acquisition time of 22.5 seconds for a data rate of 174 bps. At less utility (higher energy contrast) the acquisition time can be reduced to 7.5 seconds, corresponding to 5 telemetry frames. The synchronization code consumes less than 6.4 percent of the available power.

The effective bandwidth for SNR measurement is specified by Saliga as twice the "symbol rate". This amounts to

about 11 KHz at a throughput of 174 bps, considering that each chip has two symbols (for clocking purposes) and each word has 127 chips. At this bandwidth a power contrast of $\sigma = -14.5$ db is required for a word error ratio of 0.1 (bit error ratio of 5.10^{-2}). This gives the system a maximum bit density (throughput) of $127/11000 = 0.01156$ (-19.4 db) and a maximum utility of $u = -19.4 + 14.5 = -4.9$ db. The signal base per dimension is smaller than for the 5-bit system ($\beta_c = 6.45$) but still large when compared with the theoretical minimum of 0.5.

L3. *Experimental Development in Other Institutions*

Though the most significant early development efforts in multidimensional nonbinary systems originated in the NASA laboratories (primarily JPL), many other institutions and industrial laboratories are now participating in the exploration of such systems. The following short chronological list may give a superficial survey of the magnitude of this effort. A reliable comparison of the results is impossible, due to the scarce data published so far.

Price and Greene (58) published the first comprehensive description of a binary communications system that used two orthogonal (or nearly orthogonal) shift register sequences as waveforms with large signal base. The system received the code name RAKE. Though the system is essentially a binary system, the waveforms themselves form a multidimensional codebook with two code words and $l = 2^k - 1$ dimensions with k being an integer. These characteristics, together with the matched filter extractor and maximum likelihood decider, make the RAKE system clearly a precursor of the systems with larger codebooks to be discussed here. The 53 references in the Price and Green (58) paper give a good cross section of the literature prior to 1958. The autocorrelation properties of the PR sequences (Campbell 59) made the RAKE system particularly suitable for transmission over multipath channels.

Hove and Lightfoot (63) contributed a paper on the design of matched filters for high chip rates using active RC circuitry. The 17 references in that paper may serve as a review of the precursor technology of LSI matched filter extractors. A Russian paper (Mitryayev 63) shows the interest of engineers in that country in systems with group codes as codebooks. Ballard (63) developed an experimental system called ORTHOMUX, which uses sets of orthogonal binary sequences as carriers in a multiplex system. Notice that the resulting composite signal forms transmission words of nonbinary elements. Lachs (63) reported a simulation of multidimensional nonbinary codebooks by a digital computer. His investigation aimed at the maximization of the minimum distance between code words in a multidimensional vector space.

In 1964 the interest in applying binary PN sequences in communications systems continued with a paper (Shepertycki 64) on their application as a random message source in telemetry error measurements. Blasbalg et al (64) presented the concept of a random access communications system that applied large sets of PN sequences in combination with a frequency offset of subcarrier frequencies as addresses for several thousand receivers that might subscribe to the participation in such a system. Computer simulation was used to explore the system performance with 2, 5, 10, and 15 equal power simultaneous talkers. Lightfoot and Kogut (64) and Lightfoot (65) described the development of an experimental data transmission terminal for HF radio frequency transmission which makes use of PR sequences in a similar mode to the RAKE system. The novel feature is a programmable matched filter consisting of 100 separate active RC circuits. The particular detection technique applied in this experimental terminal minimizes the synchronization requirements between transmitter and receiver.

A detailed description of an experimental terminal for a quantized pulse position modulation (QPPM) system is given by Huber (65). QPPM, also called MTSK (multiple time shift keying), or as in this case, *monocycle position modulation*, has become the classical example for describing a typical orthogonal nonbinary communications system (section K5 above). The report shows that systems of this kind have attractive characteristics for tactical army applications. Wang (65) found a comma-free square binary codebook for a telemetry system of the kind described at the beginning of this section. This method, which is based on results originally published by Stiffler (62), is applied to a biorthogonal codebook of $l = 16$, $n = 5$. Laboratory tests show good agreement with the theory and prove that systems with such codebooks have improved synchronization characteristics. Greim et al (65) are the authors of a large test report which was mentioned earlier. It compares the performance of two novel experimental systems which incorporate certain nonbinary features (DEFT and KATHRYN) with commercially available systems. All systems are specifically designed for operation over long distance radio links in the HF bands. The report does not describe monosignal multidimensional nonbinary systems, in which we are interested here, but may be useful in systems comparisons.

An interesting data transmission system for geophysical research is the pulse morse code modulation (PMCM) system described by Toney and Walter (65). It is a kind of hybrid system with nonbinary features. An equivalent binary codebook representing the coding modes of the system can be found. An experimental setup for the evaluation of variable rate satellite communication systems with PN sequences has been designed in Canada and is described by Hedemark (65). Also from Canada originated a brief comparison between analog matched filters and digital matched filters for the detection of PN sequences from Ionosonde transmitters (Coll et al 65).

Gold (65) wrote a paper on the generation of large families of binary sequences with specified small cross-correlation characteristics. The application of such sequences may be found in spread spectrum multiplexing systems. Motley and Melvin (66) described a higher order alphabet (HOA equal to nonbinary) system using a biorthogonal codebook of 8 dimensions and 16 code words ($n = 4$) and compared its performance over high frequency (HF) radio channels with conventional diversity systems. In 1966 also appeared a report by Gibson (66) on the performance of data transmission systems using nonbinary alphabets. We quoted this report earlier but we would like to stress now the comprehensive treatment of logical circuit design for nonbinary systems that can be found in this report. This may be useful for design engineers.

Blasbalg et al (67) used pseudo-noise binary sequences in a conceptual system design of a satellite communications system connecting aircraft via the satellite to other aircraft or to ground stations. Barnes et al (67) described the data handling system in the ground station for the reception of the telemetry signals from the biorthogonal 8-bit-per-word system that we discussed above according to the description of Saliga (67). Andrews (67) reports the design and field tests of a multipurpose acoustic telemetering system that uses PN sequences of different lengths. A long sequence serves as a timing reference and a short sequence carries data.

Concluding this section, we may state that the most advanced designs of information transmission systems with multidimensional codebooks may be found in space communications applications. Evidently, these are power limited designs and therefore operate at very low bit density. There is a large number of other conceptual designs and of some experimental systems where designers try to make use of PN sequences with binary elements. Practically no reports are available of experimental efforts in the area of codebooks with nonbinary elements or of system designs of multidimensional systems with medium or high bit density. It seems that the practical engineers have not yet discovered the theoretical conclusions which have been available for at

least ten years and which clearly prove that the most economical application of multidimensional systems will be in connection with nonbinary codebooks (linear real coding) and in the range of high bit densities. Surely the reason for this lack of following the theoretical lead is also the lack of operational coders and decoders which can handle large codebooks with nonbinary elements. The amount of circuitry required for this task will turn these coding devices into special purpose computers. Yet it seems that LSI will make such feats possible in the near future.

Part IV—Nonbinary Signal Transmission

Introduction

Part III contained a review of all codebooks for block codes with binary and with nonbinary elements. The mathematical models which we presented in Part III assumed that an ideal transmission system would be available for transferring the code words from the output of the encoder to the input of the decoder. We frequently referred to the practical difficulties that have to be overcome when attempting to come close to the specification of 2B samples per second used in such ideal transmission systems.

In the following Part IV it will be shown that there is a research effort under way which parallels the research on codebooks. This parallel research effort is known under such different names as signal theory, waveform research, research on bandlimited transmission functions, and others. It is closely related to, but not identical with, modulation theory, filter theory, and spectral analysis.

M. Fundamentals of Signal Transmission Modes

Figure 16 in Part III shows that in multidimensional systems the encoder delivers l-tuples of real numbers, which cause selections in the modulator matrix from a waveform library of q members. In special cases q may go to infinity; such special cases lead to libraries of waveforms with continuously variable parameters. If q is a finite integer we need waveform libraries with fixed parameters.

M1. *The Research on Waveform Libraries for Multidimensional Systems*

The research on waveform libraries basically follows two schools of thought.

The research effort of one school is directed to the exploration of linear and nonlinear filters (or active converters) with capabilities to deliver at the output a desirable waveform library if the individual codebook words are placed at the input of these conversion devices. McAuliffe (59) studied the impulsing of linear networks and their use both as signal generator in data transmitters and as matched filters in data receivers. Dishal (59) investigated in particular the impulsing of filters with Gaussian response shape. Eggers and Sze (61) reviewed the state-of-the-art of function generators, as they are used in analog computing technology, primarily in simulation systems. Many of these devices operate as digital-to-analog converters rather than as signal generators. The waveform characteristics are completely determined by the digital input ; the device merely shapes the signals to deliver the desired waveform at the output. Billings (62) based his investigation of a sequential generating filter on the fact that the output of a linear network is the convolution of the input signal and the impulse response of the network. His function generator delivers an approximation to the desired impulse response every time it is triggered, and its total output is the sum of the impulse responses to all input trigger pulses. Similar devices are reported (Romanov and Zelentsov 64) using a pulse filter with ferrite cores, and (Fenwick and Barry 65) producing a linear frequency sweep waveform with the help of a commercial frequency synthesizer when receiving digital input sequences. A rotational transformation of signals (Lang 63) is an analog encoding technique which will be of interest to codebook specialists and to waveform specialists. Related contributions are also available from Hofheimer and Perry (58), from DeClaris et al (61), and from Fulford (64).

The research of the other school was guided by the necessity to overcome the practical limitations of transmitters and channels mentioned in Part I, subsections B1 and B2. Many of the contributors mentioned in those subsections also participated in research efforts to overcome these limitations. In addition, Tufts (65) derived optimum transmitter pulse shapes and carried out the joint optimization of transmitter and receiver in detail for the case of signalling through a noisy RC filter. Although investigation is primarily for PAM, it is also of fundamental importance for nonbinary systems. Pushman (63) calculated the spectral density distributions of signals for binary data transmission under rather practical assumptions.

More complex waveforms will be synthesized from orthogonal components. For the low-pass channel it is convenient to use $\sin w_o t / w_o t$ components shifted in time for increments of $\Delta t = \pi / w_o$, where $w_o = 2\pi f_o$ is the cutoff frequency of the low-pass channel (ideally assumed to have a rectangular frequency response). Zayezdnyy and Eydukyavichyus (63) developed a system of abbreviated representation which they demonstrate with the help of numerical examples. For many signals of length T it is possible to use less than $2f_o T$ component waveforms, if a new smaller set of component waveforms can be found through transformations of the $\sin w_o t / w_o t$ set of $2f_o T$ components ($f_o = w_o / 2\pi$). The two Russian authors demonstrate how such transformations can be found.

Arthurs and Dym (62) prove a theorem that any finite set of physically realizable waveforms of duration T, such as $s_1(t), s_2(t), \ldots, s_q(t)$, may be expressed as a linear combination of l orthonormal waveforms $\phi_1(t)$; $\phi_2(t)$; . . . ; $\phi_l(t)$ with $l \leqslant q$:

$$\begin{aligned} s_1(t) &= a_{11}\phi_1(t) + a_{12}\phi_2(t) + \ldots + a_{1l}\phi_l(t) \\ s_2(t) &= a_{12}\phi_1(t) + a_{22}\phi_2(t) + \ldots + a_{2l}\phi_l(t) \\ &\vdots \\ s_q(t) &= a_q\phi_1(t) + a_{q2}\phi_2(t) + \ldots + a_{ql}\phi_l(t) \end{aligned} \tag{93}$$

where the a_{ij} are real numbers given by the formula :

$$a_{ij} = \int_0^T s_i(t) \cdot \phi_j(t) dt \quad , \tag{94}$$

with

$$\int_0^T \phi_i(t) \cdot \phi_j(t) dt = \begin{matrix} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{matrix} \tag{95}$$

There is evident similarity between this representation of waveforms by orthonormal component functions and the representation of code words by the components of an l-dimensional space. This apparently is the strongest link between the codebook of the encoder matrix and the waveform library of the modulator matrix in the basic block diagram of a nonbinary system shown in fig. 16.

Yet, in describing systems, there are not many publications which make full use of this similarity. Part II explained how this general waveform library can be specialized to describe the one-dimensional nonbinary systems.

M2. *The Early History of the Waveform Approach*

The *waveform approach* originated in the work of Nyquist published in 1924 and in 1928. The result of these early investigations was the general recognition of the Nyquist rate, 2B, as the highest possible rate at which independent elementary signals can be transmitted without mutual interference over an ideal low-pass channel of bandwidth B. Nyquist's later research led to a paper (Nyquist and Pfleger 40) which anticipates the later highly important concept of analytical signals. This is followed by Gabor's theory of communications (Gabor 46) that preceded Shannon's mathematical theory of communication by two years. Gabor's emphasis was more on deterministic signals and on waveforms than on random parameters and on probability measures. He may be considered as the father of the signal theory (waveform approach) as much as Shannon may be considered the father of the statistical communication theory that led to the coding approach of the 1950's. Around the middle of the century a number of books summarized the state of the fundamental theories of communications ; they best serve *in toto* as a record of the signal theory of the time :

(i) 1948 : S. Goldman, " Frequency Analysis, Modulation, and Noise ", New York, McGraw-Hill Book Co., a theory of waveforms and modulated signals on the basis of Fourier series and Fourier integrals.

(ii) 1949 : K. Küpfmüller, " Die Systemtheorie der elektrischen Nachrichtenübertragung ", Stuttgart, S. Hirzel, is a textbook in German language which approaches the signal theory from the well-established theory of transients in electrical circuits.

(iii) 1949 : C. Cherry, " Pulses and Transients in Communications Circuits ", London, Chapman and Hall, Ltd., is particularly famous for the extensive treatment of the spectra of pulsive waveforms and of the transient analysis of networks including lines.

(iv) 1950 : J. L. Lawson and G. E. Uhlenbeck (eds.), " Threshold Signals ", Vol. 24 of the Radiation Laboratory Series, New York, McGraw-Hill Book Co., is now considered as a classical textbook for weak signal detection theory. It approaches signal theory from the radar engineer's point of view.

(v) 1953 : P. M. Woodward, " Probability and Information Theory with Applications to Radar ", London, Pergamon Press, Ltd., discloses many wartime results of the famous Radar and Telecommunications Research Establishment, Malvern, England. Some of the mathematical results of this book formed the basis of extended research work in the United States.

(vi) 1953 : D. A. Bell, " Information Theory and Its Engineering Applications ", explains in nonmathematical language how the early signal theory of Lord Kelvin, Nyquist, Hartley, and Gabor may be integrated into the more modern information theory of Wiener and Shannon.

(vii) 1953 : S. Goldman, " Information Theory ", London, Constable and Co., Ltd., has an introduction to the use of signal space, and chapters on the information theory aspects of modulation and on linear correlation.

Despite this impressive array of textbooks, the waveform approach to the conceptual design of multidimensional data transmission systems made only marginal progress prior to 1957. Interesting results came from Sunde (54) and from French researchers promoting the theory of analytical signals (Ville 48, Oswald 50 and 51). The French researchers put the signal theory of the bandpass channel on a sound mathematical basis by combining the two quadrature-phase components of a real signal into a complex " analytical signal ". By this method it is possible to apply the complex time domain. When relating this domain to the well-known complex frequency domain, one can apply the theory of analytical functions. The work of Sunde combines signal theory with circuit theory by giving the general relations between frequency response and pulse transmission characteristics in the form of a compendium for engineering applications. Some of the general results of signal theory prior to 1957 were summarized in a special issue of " IRE Transactions of Circuit Theory ", Dec. 1956, with an introductory paper by Huggins (56). Although few of these results are directly applicable to waveform alphabets of multidimensional data systems, many of the papers of that special issue will have a strong indirect influence on waveform alphabets in the future.

M3. *Progress of the Waveform Approach Since* 1956

In this section we shall review only those contributions to waveform alphabets that are of a general nature. The evolution of specific sets of waveforms will be discussed in the next two sections.

The special signal theory issue of the " IRE Trans. on Circuit Theory " (Huggins 56) marked the starting point for a more intensive research on waveforms for data transmission. In that issue we find an article by Oswald (56) extending his earlier theory of analytical signals to the case of bandlimited signals and applying this extended theory to the analysis and synthesis of waveforms in carrier systems. While the older signal theory of carrier systems, using the Fourier integral, relied completely on the study of the transmission of sinusoidal waves, Oswald (56) showed that the concept of analytical signals permits the extension of the signal theory to cases of arbitrary signals. This is achieved through the introduction of a quadrature signal. The extended theory defined also an *instantaneous frequency* that played an essential role in French research contributions on message intelligibility.

Oswald's paper, together with a paper by Lampard (56b), in the same special issue gave a fundamental discussion of the meaning of band limitation as related to the uncertainty principles. Despite the existence of Gabor's definitions of an " effective temporal duration " and an " effective spectral width " (Gabor 46), a stronger expression of the time-bandwidth relation was felt desirable at that time. (These, and later, results are discussed in the next subsection, M4.)

In 1958 several authors amplified the results of the special signal theory issue of 1956. Silverman (58) pointed out some conditions for the existence of moments of analytical signals which had been generally overlooked. Dugundji (58) demonstrated that the Hilbert transform of a waveform can be used profitably in the analysis of rather arbitrary signals which had been introduced by Rice (44) as multichromatic waveforms. Dugundji (on page 53) gives a formula for the envelope of a carrier-band waveform which is " much simpler than Rice's formula and which is easier to handle analytically ". Dugundji uses a complex valued function called the pre-envelope and he shows that Rice's envelope is the absolute value of the pre-envelope. " By using the pre-envelope the envelope of the output of a linear filter is easily calculated and this is used to compute the first probability density for the envelope of the output of an arbitrary linear filter when the input is an arbitrary signal plus Gaussian noise ".

Also in 1958, Wernikoff submitted his doctoral thesis at MIT under the title : " A Theory of Signals ". Wernikoff developed a formalism for the analysis of signals and their interaction with linear systems. Signals are considered as vectors in a space that is not necessarily finite dimensional. The space can span over an orthogonal basis or over a non-orthogonal (but linearly independent) set of vectors as basis. Wernikoff describes the application of his theory of signals

to the discrimination of individual events in noisy radar-like systems and to the derivation of the optimum (in the sense of least peak error) predicting filter. Beyond these examples one can see applications in communications systems with signal sets which are synthesized from orthogonal components.

Another major contribution to the research on signal transmission modes is more on the practical side : Blackman and Tukey (58) contributed a major paper on the measurement of power spectra from the point of view of communications engineering. This paper, in two parts, has since become a classic. The authors successfully combined the ideas of transmission theory (problems of instrumentation, recording, and analysis) with the ideas of statistical estimation theory. This combination provides the insight necessary for a sound theory of data acquisition and data reduction. Such a theory is essential for the optimum extraction of the noisy information-carrying waveforms in a receiver of a nonbinary transmission system.

Several auxiliary methods with important applications in signal transmission systems were either reviewed or additionally developed in the years 1959 to 1962. Helm (59) devoted a fundamental paper to the Z-transform, particularly to its use in real-time systems controlled by digital computers. Such problems will be of high importance for the operation of multidimensional information transmission systems of high dimensionality ($l > 1000$). Fano (61), in chapter 5.7, discusses the fundamentals of bandlimited time functions and forms a bridge between the coding approach and the waveform approach. The importance of the Hilbert transform for the signal transmission theory has been stressed above in connection with the discussion of analytical signals. Kuo and Freeny (62) reviewed in a tutorial paper the properties of this transform, showed its application for single sideband modulation, and discussed its connection with the concept of the pre-envelope. Beutler (61) showed that the Hilbert space concept could lead to a unified approach to sampling theorems for—in the wide sense—stationary random processes. Papoulis (62) had some interesting comments on an estimate of the truncation error in Fourier integrals. Kurz (62) analyzed an interesting method for the generation of a waveform alphabet. He called his method *orthogonal digit coding*. The signal set (average power limited) is formed from weighted sums of eigenfunctions generated by an integral equation with its kernel corresponding to the inverse Fourier transform of the noise power density spectrum. Specific performance results are given for several orthogonal digit codes (with nonbinary elements) when the demodulated Gaussian noise power density spectrum increases with increasing frequency. Bedrosian (62) uses the analytic signal concept to postulate a rather general formulation of all analog modulation methods. Known types of modulation are readily identified as special cases. A new type of modulation (SSB-FM), compatible with existing FM receivers, resulted from this investigation. Rubin and DiFranco (63) pointed out that the analytic signal representation of wide-band carrier signals is not completely identical with the conventional representation, which uses a variable amplitude term and a variable phase term in a trigonometric function. The difference between the two representations becomes negligible if the carrier-band signals are bandlimited.

In 1963 we notice an increasing awareness of the importance of orthogonal signal sets, either to be used directly as the waveform alphabets (Harmuth 63a), or to be used as coordinates in a signal space of high dimensionality (Harmuth 63b). Tung (63) contributed a short note to the analysis of such bandlimited signal sets, restricting his investigation to functions that could be generated from second-order ordinary linear differential equations. Tung used a minimum RMS difference criterion in his search for an optimum set of bandlimited functions that could be used as coordinates for the decomposition of any arbitrary time-function limited to the same band as the set. Dollard (63) provided a more general theory of the function space of a fixed number of dimensions when imposing time and bandwidth limitations. Dollard discussed orthogonal, biorthogonal, and simplex configurations in terms of bandlimited functions.

The analytical treatment of sets of bandlimited functions elicited more interest in 1965, with papers on a variety of important problems. Nuttall (65) derived expressions for signal sets with the minimum Gabor bandwidth when the signals themselves are abruptly time limited. Zakai (65) extended the definition of bandlimited functions and of bandlimited random processes to include functions and processes which do not possess a Fourier integral representation. The sampling theorem for such functions was derived. Azar (65), in a paper on Z-transforms, took into account a finite sampling time and showed how to make practical use of it. Yao and Thomas (65) investigated the error that occurs when a bandlimited signal is represented by a finite number of sampling terms in place of the theoretically necessary infinite number of terms. Gonsalves (65a) proved the existence of sets of real functions whose autocorrelation functions are identical with the functions themselves when taken for positive arguments. Such functions are called *autocorrelation-invariant functions*. Evidently, they will be of great importance in transmission signal analysis. The Laguerre and Legendre functions of the first kind show this property (see subsection N2).

Voelcker (66a, b) took the first step toward a unified theory of modulation. His paper, presented in two parts, summarized most of the important results of the conventional modulation theory based on analog sinusoidal signals (62 references). Voelcker developed equations for the description of simultaneously phase-modulated and envelope-modulated waveforms which are also bandlimited. The new concept is the use of real and complex zeros of the wave. It is designated as the *concept of modulation as zero manipulation*. Both kinds of zeros are mathematically interpreted in terms of the factorization of a Fourier series. Examples illustrate the relationships between zeros, spectra, envelopes, and phase functions.

The impact of a report issued in 1966 by the US Department of Commerce (66), Telecommunication Science Panel of the Commerce Technical Advisory Board, will be felt during the next two decades. The report concluded (p. 39) that " The United States has presently a grossly inadequate technical program of improving the overall effectiveness of the utilization of the spectrum." One consequence of this report will be an intensified research effort in signal theory, with the aim to provide strictly bandlimited transmission signals so that mutual interference between adjacent transmission bands can be significantly reduced.

Improved tools for analyzing and synthesizing bandlimited signals are provided by Papoulis (67) and by a new computational procedure called the *Fast Fourier Transform* (Bogert 67).

M4. *Fundamentals of Bandlimited Waveforms*

Multidimensional systems need transmission facilities which can keep the coordinate values of the different dimensions perfectly separated during the transmission processes. Obviously, this can be done by transmitting the dimensions in time sequence (time division) or simultaneously in different subchannels (frequency division). It has been known for many decades that it is impossible to limit a signal simultaneously with absolute precision in the time domain *and* the frequency domain. Wozencraft and Jacobs (65) proved (in Appendix 5B) that any real-time function that is identically zero over any interval of non-zero length has an infinite frequency spectrum. Conversely, it may be stated that an exactly bandlimited function (i.e., a function with a frequency spectrum that is identically zero outside a finite frequency band) must have an identically zero value over all times as soon as it is identically zero over any finite time interval.

This fundamental theorem for time-limited or bandlimited functions caused early reaserch workers in signal theory to search for a quantitative answer to the question for the smallest time-bandwidth product that any signal of finite energy ever can assume. Evidently the difficulty is in the definition of an " effective signal duration " and " effective signal bandwidth ". Contributions by Gabor (46), Woodward (51), Lampard (56b), and more recently by Slepian and Pollak (61), and Landau and Pollak (61 and 62) led to results which have been summarized (and supported by analytical proofs) in the book of Wozencraft and Jacobs (65) (Appendix 5A) as follows :

There are two theorems on the time-bandwidth (TB) product of time-limited and bandlimited waveforms. The one theorem is due to Landau and Pollak (62), and the other is due to Shannon. Both theorems prove that a low-pass time function of unit energy that is exactly time-limited to an interval $-T/2 < t < +T/2$ and that has not more than a fraction (θ) of the total energy outside a low-pass band from O to B Hz can be closely approximated by a set of L orthonormal functions. The theorems give both the number L and the closeness of approximation according to an RMS criterion. The first theorem is the stronger one ; it requires that no linear combination of all orthonormal functions must ever have more than a fraction (θ) of the energy outside the frequency band O to B. The second theorem (Shannon) requires that merely the orthonormal functions themselves must meet the above condition, while certain linear combination of orthonormal functions could have a larger fraction than θ outside the frequency band O to B. With these restraints imposed, the following formulas may be applied :

First Theorem : (Landau and Pollak, stronger constraint) :

$$L = \text{largest integer} \leqslant 2TB + 1 \tag{96a}$$

$$\epsilon_{\Delta} = 12\,\theta \tag{96b}$$

Second Theorem : (Shannon, weaker constraint) :

$$L = \text{largest integer} \leqslant 2TB + \frac{12}{\Delta}\left(1 + \frac{1}{\pi^2}\log_e 2TB\right) \tag{97a}$$

$$\epsilon_{\Delta} = \frac{12\,\theta}{12 - \Delta}; \quad \text{for all } \Delta,\ 0 < \Delta < 12 \tag{97b}$$

In both cases the error of the approximation (ϵ_{Δ}) is defined as the energy of the difference between the low-pass time function, f(t), and its approximation :

$$\epsilon_{\Delta} < \int_{-\infty}^{+\infty}\left[f(t) - \sum_{i=1}^{L}\lambda_i\,\psi_i(t)\right]^2 dt \quad , \tag{98a}$$

with

$$\lambda_i = \int_{-\infty}^{+\infty} f(t)\,\psi_i(t)\,dt \tag{98b}$$

Both theorems can easily be expanded to bandpass waveforms. Evidently the best approximation will be achieved by that set of orthonormal functions $\{\psi_i(t)\}$ which can supply the largest number of functions $i = 1, 2, \ldots, q_o$, meeting the condition of the band limitation (θ for any given bandwidth B). Thus, the theorems may also be used to predict the size of the best orthonormal set of a given TB product when meeting the θ fraction restraints of each theorem.

This leads to the following equations for q_o :

First Theorem : (stronger constraint) :

$$q_o \leqslant L \leqslant 2TB + 1 \tag{99}$$

Second Theorem :

$$q_o < 2TB \cdot \frac{12 - \Delta}{12(1 - \eta) - \Delta}\left[1 + \frac{12}{2TB \cdot \Delta}\left(1 + \frac{1}{2}\log_e 2TB\right)\right] \tag{100}$$

for all Δ, $0 < \Delta < 12$.

For $T \to \infty$ there is a limiting value :

$$\lim_{T \to \infty} \frac{q_o}{T} \leqslant \frac{2B}{1 - \theta} \tag{101}$$

Equation 101 leads to the well-known Nyquist rate of independent samples (dimensions) when the exactly band-limited sin x over x waveforms for which $\theta = 0$ are used.

For a very large TB product (for example, when $2TB \geq 100$) and for $\theta = 1/12$, equation 100 gives the frequently used result :

$$q_o < 2.4\ TB \tag{102}$$

The above equations will permit future comparisons of the band limitation of the various sets of orthonormal waveforms that have been suggested in the last decade. Unfortunately, for many of these sets, θ has not been computed yet as a function of a suitably normalized bandwidth.

Contributions closely related to the TB problem of band-limited waveforms are also available from Petrich (63), Dollard (63), Hofstetter (64), Tufts and Shnidman (64), DiToro and Steiglitz (65), Jagerman (65), Nuttall (65), and Papoulis (67).

M5. *Concepts of Systems with Universal Signal Space*

There have been numerous efforts to combine the coding approach and the waveform approach to integrated systems. An integrated system may be defined as a system where each of the separate subsystems is first optimized according to its own performance criteria and where easily accessible operational parameters of each subsystem are modified until the total system reaches its optimum performance under the externally imposed restraints (cost, peak or average power limitations, reliability, serviceability, etc.). The different functions (subsystems) in a communications system that may be subjected to such an optimization process were reviewed as early as 1959 (Filipowsky 59). Here we are concerned only with the integration of the two subsystems shown in fig. 16 : the digital encoder and the analog waveform generator (modulator). The optimal combination of coding and modulation has been the subject of several conceptual expositions during the time prior to 1960 (Costas 52, 56, 59). A comprehensive study was performed at the Bell Telephone Laboratories (Dollard et al 61 and I. Jacobs et al 63), summarizing the research results at that time, laying the groundwork for improved system designs, and giving an equal importance to coding and to modulation methods. Kirshner (62) reviewed the theory of a multidimensional signal space for the purpose of increasing the number of available channels by allowing a certain amount of mutual interference between the channels. Halsted (63) applied the signal space concept to show how square coding matrices could be used to synthesize optimum signal ensembles for nonlinear and non-Gaussian channels. In a similar way it seems possible to counteract pulsive noise by a rotational transformation of signals (Lang 63). This concept is related to the smearing techniques of continuous systems and its application allows interesting explorations of the characteristics of special codes. This is particularly true when applying nonorthogonal transformations in place of the pure rotational transformation. Dollard (63) developed the concept of a universal signal space on a mathematically rigorous basis and Amari (65) penetrated more deeply into the metrics of signal spaces. He defined the entropy of a signal space and explored the problem of continuous mapping between signal spaces of different dimensions.

H. D. Luke (65) continued the approach which had been initiated by McAuliffe (59), applying the later results of signal theory. By generating special pulse sequences in the encoder, Luke generated a suitable waveform alphabet in transversal filters. This method is a practical approach toward a system with universal signal space. Luke's results are related to an effort by S. H. Chang (66a) (section

K2). Chang is primarily interested in the generation of nonbinary orthogonal codes with integer elements ; that part is related to Luke's approach. Chang, however, does not use the output of the encoder as the input sequence for pulsing a network to generate a waveform.

An important stepping stone to the further evolution of the network pulsing techniques is the investigation of the frequency spectra of various sets of binary and nonbinary sequences. Gorog (68) performed such an investigation on various redundant codes. When also incorporating the channel characteristics, primarily the frequency response, into this integrated optimization process of codes and wave shaping devices, one follows closely the conceptual approach which Costas suggested in 1952. A more recent study in this direction is due to Hancock and Quincy (66).

Possibly the most fundamental contribution to the theory of a universal signal space is the paper by Wyner (66b) about the capacity of the bandlimited Gaussian channel. Wyner claims that the famous channel capacity formula of Shannon (equations 17 to 23 and curve 1 in most of the utility charts of this paper) never has been justified by a coding theorem (and converse). Such a theorem is necessary to establish Shannon's channel capacity as the supreme expression for the capacity of more restrictive mathematical models. Wyner proposes a number of physically consistent models and establishes equation 17 as the capacity for each of the models.

N. Special Waveform Alphabets

N1. *Monary and Binary Large TB Waveforms*

Radar technology was the first to make extensive use of the matched filter technique as originally suggested by Wiener (49) and Woodward (53). This matched filter technique is an optimum method to extract a known signal from noise. The technique found conceptual acceptance in communications with the first mathematical model of an ideal binary system (Reiger 53). One of the first practical matched filter extractors was applied in the RAKE system (Price and Green 58, section L3). Sussman (60) describes the matched filter action in that system.

Evidently the transmission waveforms which had been selected as large TB waveforms for the first binary matched filter communications systems were a good basis for the design of larger waveform alphabets. The use of such larger alphabets was clearly suggested in the interest of transmitting more information (nonbinary systems), of higher security (code switching), or for the purpose of addressing a larger number of receivers (multiple access systems).

In this connection we find that Key et al (61) discussed a method for designing large TB signals with independently specified envelope and autocorrelation function. Grettenberg (63) proposed a signal selection procedure based on a *maximum divergence criterion* which requires that the smallest value of the divergence between two distinct messages be maximized. Kadota (64) wrote about systems with random signals having large TB products and known covariance functions. Such systems constitute a mathematical idealization of some radio communication links containing a random medium.

A particularly interesting class of large TB signals is the swept frequency signals, also called chirp signals because of their similarity with the audio waveform of a siren sound. Such signals change their instantaneous frequency, usually in a linear manner, during the signal duration. A binary system can, of course, use one waveform starting with the lowest frequency in the band and ending with the highest frequency and another waveform following the opposite pattern. When using more complex patterns of frequency variation, larger sets of waveforms may be designed. Signals of this kind have been used in radar systems for some time, but their application in communications is of more recent date. Holland-Moritz et al (66) described a communications system which uses *swept frequency modulation*, with the individual large TB waveforms overlapping in the time domain. This feature makes the system a polysignal system, following the definitions used in this paper. Such systems are beyond the scope of the present paper ; yet the description of the waveforms, of their modes of generation, and of the extraction devices may be of interest here. The description of another communications system with FM chirp signals is due to Griffiths and Smith (67).

N2. *Polynomial Waveform Alphabets*

The following review will show that there is much interest in orthogonal polynomials and in their use as waveform alphabets, primarily as orthonormal alphabets. Yet it is amazing that the mathematical foundations for such efforts are at least one century old. The particular orthogonal polynomials that gained most of the interest of communications engineers are (in the order of their frequency of application in the communications literature) :

(i) *The Legendre Polynomials.* Andrien Marie Legendre (1752-1833), a French mathematician, is noted for his work on the theory of numbers (" Théorie des Nombres ", 1830) and for his extensive study of elliptic integrals (" Traité des Fonctions Elliptiques ", 3 vols., 1825-1832).

(ii) *The Laguerre Polynomials.* Edmond Niclas Laguerre (1834-1886), a French mathematician, is famous for his contributions to modern geometry. His collected works were published in 1898.

(iii) *The Hermite Polynomials.* Charles Hermite (1822-1901), a French mathematician, made valuable contributions to the theory of numbers and the theory of elliptic functions.

(iv) *The Jacobi Polynomials.* Karl Gustav Jakob Jacobi (1804-1851), German mathematician, is noted for his work on elliptic functions described in his " Fundamenta Nova Theoriae Functionum Ellipticarum " (1829) and for his contributions to the theory of numbers.

Orthogonal polynomials are sets of functions $[f_n(x)]$ of degree n defined on the interval $a \le x \le b$ with respect to the weight function $\omega(x)$. Their orthogonality property can be expressed in the following form (Abramowitz and Stegun 64) :

$$\int_a^b \omega(x) f_n(x) \cdot f_m(x) dx = 0 \tag{103}$$

$(n \neq m ;\ n, m = 0, 1, 2, \ldots)$

The weight function $\omega(x)$ has non-negative values for all x values. The polynomials $f_i(x)$ need to be " standardized " ; i.e., they have to meet the condition of equation 104 so that they may be in the correct form to satisfy a number of relationships that apply to all the orthogonal polynomials mentioned above and to many other kinds of orthogonal polynomials which may have potential applications in communications.

$$\int_a^b \omega(x) f_n^2(x) dx = h_n \tag{104}$$

$$f_n(x) = k_n x^n + k'_n x^{n-1} + \ldots$$

$(n = 0, 1, 2, \ldots)$

The relationships that all the classes of orthogonal polynomials must satisfy are listed below.

(i) All polynomials are solutions of the general second order differential equation :

$$g_2(x) f''_n + g_1(x) f'_n + g_0(n) f_n = 0 \quad , \tag{105}$$

where $g_2(x)$ and $g_1(x)$ are independent of n but depend on the class of polynomials. $g_0(n)$ depends only on n and not on x, but differs for the various classes of polynomials. Abramowitz and Stegun (64) pub-

lished tables of these functions for all orthogonal polynomials. Table 1 gives these functions and k_n (equation 106a) and h_n (equations 104, 106a) for the polynomials of interest in eonnection with the present paper.

(ii) All polynomials satisfy the following recurrence relation :

$$f_{n+1}(x) = (a_n + b_n x) \cdot f_n(x) - c_n f_{n-1}(x) ; \qquad (106)$$

$$b_n = \frac{k_n + 1}{k_n}; \quad a_n = b_n \left(\frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n}\right);$$

$$c_n = \frac{k_{n+1} \cdot k_{n-1} \cdot h_n}{k_n^2 \cdot h_{n-1}} \qquad (106a)$$

(iii) All polynomials satisfy the Rodrigues formula :

$$f_n(x) = \frac{1}{\alpha_n(n) \cdot \rho(x)} \frac{d^n}{dx^n} \{\rho(x) \cdot [g(x)]^n\} \qquad (107)$$

The functions $\alpha_n(n)$, $\rho(x)$, and $g(x)$ are listed in Abramowitz and Stegun (64) for each class of orthogonal polynomials. Table 2 gives these functions for the four classes of orthogonal polynomials of special interest in the present paper.

(iv) For some of the polynomials there are representations in terms of hypergeometric functions :

$$f_n(x) = d_n \cdot F(a ; \ b ; \ c ; \ \varkappa(x)) \qquad (108)$$

Table 2 gives the special values for d_n, a, b, c, and k(x) for two classes of orthogonal polynomials of interest in this paper.

(v) There are many important interrelations between orthogonal polynomials of the same class. The reader is referred to Abramowitz and Stegun (64), to Jonsone (59), to Gernimus (61), and to Boas, Jr. and Buck (64) for further details.

In many publications, reference is made to orthogonal functions, though there is no well-defined terminology clarifying the difference between orthogonal polynomials and orthogonal functions. Possibly the best explanation for electronic engineers is given in Y. W. Lee (60) who describes in full detail the orthonormalization process for Laguerre functions and Legendre functions when starting with the respective orthogonal polynomials described in Tables 1 and 2. From this and similar presentations one may conclude that the general practice is to call the product of an orthogonal polynomial with the root of the weighting function, the orthogonal function of the particular class. In addition, Y. W. Lee (60) shows how to apply orthogonal polynomials and orthogonal functions in the complex domain.

Table 1

Coefficient functions and other parameters for various classes of orthogonal polynomials

Class of Polynomials	Symbol	$g_2(x)$	$g_1(x)$	$g_o(n)$	k_n	h_n
Legendre	$P_n(x)$	$1 - x^2$	$-2x$	$n(n+1)$	$\frac{(2n)!}{2^n(n!)^2}$	$\frac{2}{2n+1}$
Laguerre	$L_n(x)$	x	$1 - x$	n	$\frac{(-1)^n}{n!}$	1
Hermite	$H_n(x)$	1	$-2x$	$2n$	2^n	$\sqrt{\pi} \cdot 2^n \cdot n!$
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$1 - x^2$	$\beta - \alpha - (\alpha + \beta + 2)x$	$n(n + \alpha + \beta + 1)$	$\frac{1}{2^n}\binom{2n + \alpha + \beta}{n}$	$\frac{2^{\alpha+\beta+1}}{2n + \alpha + \beta + 1} \cdot \frac{\Gamma(n + \alpha + 1) \cdot \Gamma(n + \beta + 1)}{n!\Gamma(n + \alpha + \beta + 1)}$

Table 2

Coefficient functions and other parameters for the Rodrigues' formula and the hypergeometric function when representing various classes of orthogonal polynomials

Class of Polynomials	Symbol	$\alpha_n(n)$	$\rho(x)$	$g(x)$	d_n	a	b	c	$\varkappa(x)$
Legendre	$P_n(x)$	$(-1)^n \cdot 2^n \cdot n!$	1	$1 - x^2$	1	$-n$	$n + 1$	1	$\frac{1-x}{2}$
Laguerre	$L_n(x)$	$n!$	e^{-x}	x	—	—	—	—	—
Hermite	$H_n(x)$	$(-1)^n$	e^{-x^2}	1	—	—	—	—	—
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$(-1)^n \cdot 2^n \cdot n!$	$(1-x)^\alpha \cdot (1+x)^\beta$	$1 - x^2$	$\binom{n+\alpha}{n}$	$-n$	$n + \alpha + \beta + 1$	$\alpha + 1$	$\frac{1-x}{2}$

The use of orthogonal polynomials and functions for signal analysis was recommended by Park and Glaser (58). That report also contains the construction of a laboratory model of a signal analyzer and deals with Legendre polynomials, Laguerre functions, and Hermite functions. Forsyth (57) and Crouse (63) applied orthogonal polynomials to the problem of data fitting (with a digital computer) and to curve fitting respectively. Gonsalves (64) showed that both Laguerre and Legendre functions of the first kind have the AI (autocorrelation invariant) property. Lipkin (66) demonstrated that Russian scientists are well aware of the potential of Laguerre and Legendre polynomials for applications in communications waveform alphabets. The Rand Corporation investigated the computational techniques resulting from the application of polynomials in airborne radar signal analysis (Liechenstein 66).

Turning now to applications of specific orthogonal polynomials, we recognize that *Legendre polynomials* are frequently suggested for signal transmission and for signal processing applications. Ballard (62a) described an experimental design of a four-channel multiplex system using orthogonal Legendre polynomials as subcarrier waveforms. The system became known as the *Orthomux telemetry system.* Ballard (62b) applied the same technique of orthogonal Legendre polynomials to a real-time function analyzer-synthesizer. Garfinkel (64) published a new addition theorem for the derivative of a Legendre polynomial. Duckworth and Smith (65) discussed a problem with the generating functions of Legendre polynomials. Broome and Dean (64) presented a method for encoding complicated transient signals of seismic P-waves into a few numbers with the help of sets of orthonormal polynomials.

Laguerre polynomials or Laguerre functions were suggested by Kantz (54) for an improved method of transient synthesis, a method that was also used in 1964 for the analysis of seismic waves by Broome and Dean (64) as reported above. Some recent mathematical contributions to the theory of Laguerre polynomials are due to Head (56), Mishkin (59), Gillis and Weiss (60), Samsonenko and Ramanenko (63) and Clowes (65). Tables of Laguerre polynomials and functions are available by Flinn and Dean (64) and by Aizenshtadt et al (66). Applications of Laguerre polynomials are reported for experimental seismic equipment (Broome and Dean (64) and Dean (64)) and also for signal sorters (Mattachione (65), Gonsalves (65b) and correlators (Schetzen (64), Eier and Weinrichter (68)).

Hermite polynomials and their history were reviewed by Cameron and Martin (47). They were tabulated by Berlyand et al (62). They were used in an experimental orthogonal multiplexed communications system (Karp and Higuchi 63, Karp and Rampacek 64). The modified system has no discontinuities in the waveform and has a small TB product.

Jacobi polynomials were used by Armstrong (57 and 59) for the representation of transients in an orthogonal system.

Chebyshev polynomials are another kind of orthogonal polynomials well-known in filter theory but so far seldom used as waveform alphabets. Kulya (63 and 64) mentioned their use in an experimental vocoder system in Russia. C. S. Chang (66) submitted a proof about the discrete orthogonal property of Chebyshev polynomials which made them attractive in curve-fitting applications.

Concluding the section on polynomial waveform alphabets, we may state that orthogonal polynomials, particularly when used in modified form with optimal weighting functions, have a good potential for applications in nonbinary communications systems. Yet up to this time their applications were primarily in circuit analysis, for the decomposition of transients, and in some multiplex systems. No report, to our knowledge, has come of an experimental nonbinary monosignal system using such sets of waveforms. A comparative analysis of their bandwidth occupancy would be necessary before the relative merits of the various classes can be evaluated quantatively.

N3. *Trigonometric and Exponential Waveform Alphabets*

Huggins (56) gave a comprehensive review of the state of the signal theory at that time. Many readers who are not familiar with the special signal theory issue of the " IRE Transactions on Circuit Theory " (Dec. 1956) will be interested in the foresight of the early research workers in this special branch. It seems to this author that many excellent ideas of the time prior to 1956 have not been sufficiently exploited, even now, twelve years after their publication. One idea was the exponential waveform alphabets. Huggins (56) stressed their fundamental character. The orthogonal polynomials discussed in the previous subsection N2 use exponential weighting functions, thereby becoming carriers for exponential waveforms. The trigonometric product waveforms to be discussed in the next section surely must be related to exponential waveforms with complex exponents, although this aspect is still a subject of future exploration.

Clearly recognizing the central importance of exponential waveform alphabets, Huggins and his school of thought performed wide-ranging research. Huggins (63) summarized seven years of research at Johns Hopkins University on the representation and analysis of signals. This summary reproduces reprints of several publications which document the research results, and includes abstracts of all technical reports produced under the program. Exponential waveform alphabets, primarily sets of *orthonormal exponentials* are at the centre of all these studies. Ross (62) reviewed the most important characteristics of orthonormal exponentials and quoted four doctoral theses which investigated related subjects. Lory et al (59), Young and Huggins (62a and 62b) and McDonough (63) are among the particularly characteristic publications of this group of researchers.

Outside the research group of Johns Hopkins University we find interest in damped oscillatory exponentials (Dolansky 60) as base signals in speech signal analysis. Mendel (63) studied the problem of characterizing overdamped systems from data in the time domain by decomposition of the data with the help of exponential base functions. The results of this study were further extended to a signal approximation with non-uniformly weighted orthonormal overdamped exponentials as base vectors (Mendel 64a). Mendel (64b) also extended Armstrong's earlier work (57 and 59) by investigating the inversion of Laplace transforms by means of truncated series of orthonormal exponential functions.

Trigonometric functions as sets of orthogonal functions are actually a special case of undamped exponentials with an imaginary exponent. To be useful as waveform alphabets in nonbinary systems, the signals have to be restricted to a finite TB product, i.e., to a given signal base. Harmuth (60a and 60b) explored the possibility of using harmonically related sine and cosine functions (restricted to integer numbers of periods) as orthogonal signal sets for radio communications and for communications over telephone circuits. Franco and Lachs (61) used orthogonal sets of trigonometric functions in a polysignal waveform library as base vectors and in a special conceptual system. These orthogonal sets restrict the set of quantized vector components to the ternary set $+1,\ 0,\ -1$. The total library of output words is an equi-energy waveform library. The Franco and Lachs (61) paper, dealing with polysignal systems, is of interest because it uses an orthonormal set of trigonometric functions. Contrary to the Franco paper, Aronstein (63) compared a single signal waveform alphabet of the Harmuth (60c) variety with binary block codes, arriving at the conclusion that trigonometric waveform alphabets are superior. Kirshner (62), in his frequently mentioned paper, likewise uses an orthogonal set of trigonometric waveforms as an example. The research of Harmuth (60a, b, c) and Franco and Lachs (61) lead to the construction of a prototype data transmission equipment for field evaluation of real time data transmission over radio frequency circuits known as the *DEFT system* (Luke 64). Sets of trigonometric waveforms are also used in a spectrum analyzer receiver described by Charles and Springett (67). Fundamental

ideas along similar lines had been published as early as 1958 in a paper of *Fourier orthogonal filters* (Blasbalg 58).

The *Orthomatch Data Transmission System* uses an envelope orthogonal set of trigonometric waveforms. We discussed this system in Part II, section G (Kuhn et al (63)) and compared it with a typical PAM-FM system. In this section it will be of interest because of its waveform alphabet.

In section O it will become evident that the trigonometric waveform alphabets discussed above have recently been identified as the first order of a much larger family of trigonometric product waveforms consisting in general of more than one trigonometric factor. The single factor waveforms which have been discussed here have the disadvantage that a large percentage of all waveforms display discontinuities at the limits of the time interval, thus causing an unnecessarily large TB product.

N4. *Perfectly Bandlimited Waveforms*

Shannon (48) used a perfectly bandlimited waveform, i.e., the waveform with a rectangular energy density spectrum, as the basis for his fundamental channel capacity theorem. This waveform has in the time domain a *function of the class sin x over x*. Numerous authors before and after Shannon's classical papers used the orthogonal set of sin x over x functions shifted for a time interval of $\Delta T = 1/2B$ against each other as the classical set of bandlimited orthogonal waveforms. Numerous versions of sampling theorems are based on the same set of orthogonal functions and the values of this function are tabulated in most collections of elementary functions. In Part I we introduced the signal base β as the TB product for waveforms that do not interfere with each other. We recognized that $\beta_c = 0.5$ is the minimum value of this signal base. This minimum value again is based on the sin x over x function with a rectangular energy density spectrum ranging from zero to B Hertz and being perfectly zero for any frequency larger than B.

The problem is more complex when considering a carrier-band channel, but pairs of quadrature signals or the concept of analytical elementary signals have solved this problem satisfactorily (Giovanni 65). The problem really becomes crucial when attempting to impose an absolute limit to the signal energy simultaneously in the frequency and the time domain. We recognized in subsection M4 that this is an impossible condition, violating the uncertainty law, and that it is therefore necessary to specify, together with time and frequency limits, a percentage of the energy that can be tolerated outside those limits. Although the energy outside the frequency band B is perfectly zero for sin x over x waveforms, one will quickly recognize that the energy outside the time limit $T = 1/2B$ is significantly larger than for any other pulsive waveform. Indeed, the time function of a sin x over x waveform reaches from $-\infty$ to $+\infty$. It is therefore essential to truncate the function in the time domain (Yao 65), Dollard (63), a process which in turn disturbs the ideal rectangular frequency spectrum. The search is thus initiated for the kind of waveform which has the best overall concentration of its energy into a given TB product with a minimum of the energy outside the limits of the signal base β_c. Many indications are that an equal spread of the energy in both domains is the best compromise, and this leads to bell-shaped (or Gaussian) signals (Gabor 46).

In this subsection, however, we are concerned with perfectly bandlimited signals. The recent research on *prolate spheroidal wavefunctions* (*PSW*) is particularly interesting in this connection. Slepian and Pollak (61) described a complete set of bandlimited functions which possess the curious property of being orthogonal over a given finite interval as well as over the interval from $-\infty$ to $+\infty$. These functions are eigenfunctions of the finite Fourier transform. Their properties make them ideally suited for the study of certain questions regarding the relationship between functions and their Fourier transforms.

The application of the theory of the PSW to a number of questions about timelimited and bandlimited signals is presented by Landau and Pollak (61). In particular the authors found the possible percentage of the energy of a finite energy signal in a finite time interval *and* a finite frequency band. These authors also applied the theory of PSW to the search for signals which do the best job of simultaneous time and frequency concentration of their energy. In the area of data transmission the authors (on p. 83) are " interested in minimizing both the tail of a pulse outside its time slot and its spectrum outside of an assigned frequency band ", recognizing that it is not possible to make both these " spillovers " arbitrarily small. The authors then state that the PSW theory " gives some information on interchannel and intersymbol interference . . . . However, the relation between time limiting and bandlimiting . . . need to be better understood ; while our general results apply, the identity of the optimal function ψ_o is not known in the case that B is a projection of the transform into . . . passband."

The third part of the publications on the theory of PSW (Landau and Pollak 62) arrived at the results on the TB product versus the size of a set of orthogonal waveforms, which we discussed in section M4.

The fourth part of the paper (Slepian 64) generalized the work of the previous parts to functions of many variables. Slepian (65b) derived some asymptotic expansions for PSW, and Slepian and Sohnenblick (65) published tables of the eigenvalues associated with PSW of zero order. Saltzberg and Kurz (64) expressed signals, in terms of PSW, which were derived under the restraint of partial and total bandwidth limitations. They show how their signal synthesis method can be used successfully to improve the performance of feedback communication links. Slepian and Pollak (61) and Landau and Pollak (61) stressed the importance of PSW in antenna synthesis problems and a correspondence by Kritikos and Dresp (64) gave further results in this direction. Tung (63) in a correspondence note related the new PSW theory to the sampling theory based on sin x over x waveforms ; Rosenstark and Kurz (66) presented a conceptual design of time limited signals which are optimized against coloured Gaussian noise with the help of PSW theory. Landgrebe and Cooper (62 and 63) used the PSW theory for two-dimensional signal representations.

The practical application of orthogonal sets of waveforms derived with the help of the PSW theory is still in its infancy, although all research workers seem to be unanimous in their praise of the fundamental importance of the new theory for signal analysis and signal synthesis. Simon and Kurz (68) used the PSW theory for the frequency domain design of signals imbedded in Gaussian noise. Yet there are some hopeful beginnings on the practical side. Saltzberg and Kurz (65) reported that they investigated waveforms consisting of the sum of a few PSW for a high bit density binary communications system requiring strictly bandlimited signals. They draw positive conclusions about the potential importance of such signals but they do not offer any suggestions for their generation. Pettit (65) used the PSW as a very efficient orthogonal set in a comparison with six other waveforms such as rectangular, triangular, trapezoidal, exponential, Gaussian, and cosine-squared. Again there is no report in that paper of an experimental evaluation of PSW waveforms.

A still broader outlook into the theory of wavefunctions is due to Rhodes (64), who investigated some problems of the double orthogonality properties of spheroidal and Mathieu functions.

Concluding the subsection on perfectly bandlimited wave forms, we recognize that the PSW had a significant impact in signal theory. There is hope that, in the coming age of computer-synthesized waveform alphabets, the practical importance of PSW will increase.

N5. *Other Sets of Waveforms*

Although the sets of waveforms created directly by filtering binary sequences and modulating them on a carrier are the most frequently used waveforms today, there are indications that they will be replaced in the future by bandlimited sets of waveforms of the classes discussed in this section and in section O. The crucial problem will be the generation of sufficiently large sets of waveforms with a large TB product. One method, which is frequently suggested, will be the synthesis by a small digital computer. Not much has been published about the operational aspects and the cost problems of such procedures, but the procedures have been applied in computer simulation in studies of various classes of bandlimited waveforms.

Assuming that the computer synthesis will be a feasible method for the generation of waveform alphabets, one can consider any set of waveforms of arbitrary complexity. The problem of finding analog function circuits that can produce the mathematical function of the waveforms would no longer apply. They would be computed, and synthesized in sampled and quantized form. Simple smoothing filters, together with analog modulators, would finalize the waveform generation in the carrier band.

Under such assumptions it is interesting to notice that Fermental (65) investigated two methods of synthesizing orthogonal waveforms with all members of a set having the same autocorrelation function. One method determines polynomials of differentiation that operate on a pulse to generate the waveforms. The other method makes use of trigonometric polynomials and is thus related to the trigonometric product waveforms to be discussed in section O.

Dubrov (61) investigated sets of Bessel functions as a waveform library and compared it with sets of Jacobi, Gegenbauer, Chebyshev, and Legendre polynomials. Unfortunately, only a summary of this interesting work has become available in the U.S.A. in translated form. Levan (66) employed the theory of linear operators on Hilbert space for a class of linear networks to show that it is possible to construct orthogonal signals in the time domain without using the conventional Gram-Schmidt procedure. As a special case this approach yields the exponential waveform alphabets discussed in subsection N3. Hofstetter (64) presented a constructive procedure for determining all possible time-limited functions having the same autocorrelation function as a given time-limited function. Specific examples, including differentiated rectangular pulses, parabolic pulses, and other pulsive waveforms with discontinuities, show how to apply this procedure. Although the method is applicable to exactly time-limited waveforms, it is of importance to bandlimited waveforms, too, since there must always be both a band limitation and a time limitation.

The above few examples may serve as an indication of the wide range of classes of functions which ultimately may come into consideration as waveform alphabets. The selection is far from exhaustive.

O. Sets of Product Waveforms

It is astounding to find out that apparently not many attempts have been made to define waveform alphabets on the basis of the products of elementary functions. Yet the simple amplitude modulation process is based on just such a mathematical relationship. It is easy to see that products evidently can define zero values of composite functions more precisely than sums of elementary functions can do. This author, some years ago, investigated modulated signals and discovered a family of bandlimited waveforms, which has since been extensively explored. These waveforms are called *trigonometric product waveforms* (*TPW*).

O1. *Definitions for Trigonometric Product Waveforms*

Trigonometric product waveforms have recently been introduced as a special kind of bandlimited signals with very interesting characteristics (Filipowsky 67a and 68b). They are defined by the following equation :

$$f_{p,k,q}(t) = A_q \prod_{i=1}^{k} \sin\left(is_i\,\omega_o t + r_i \frac{\pi}{2}\right) \tag{109}$$

$s_i = 0$ or $+1$; $r_i = 0$ or $+1$; q and p depend on the choice of s_i and r_i .

Trigonometric product waveforms may be defined on the unbounded interval $-\infty < t < +\infty$ or on the bounded interval [a, b]. Particularly useful seems the bounded interval $-\pi/2 \leq \omega_o t \leq +\pi/2$. Using that interval and substituting $x = \omega_o t$; $A_q = 1$, one can see that a typical trigonometric product waveform may have the following definition :

$$f_{4,12,347}(x) = \sin x \,.\, \cos 3x \,.\, \sin 7x \,.\, \cos 12x \tag{110}$$

The index $p = 4$, indicates the *order* of the waveform specifying the number of nontrivial trigonometric factors.

The index, $k = 12$, represents the *class* of the waveform, i.e., the highest integer used as a harmonic factor in any one of the trigonometric factors. It is a useful convention to write the product with increasing harmonics going from left to right. Notice that the defininING equation, equation 109, permits the degeneration of any trigonometric factor to a factor of the constant value " 1 ". This happens to the i-th factor when $s_i = 0$ and $r_i = +1$ are chosen. In the example of equation 110 this particular choice has been made for $i = 2, 4, 5, 6, 8, 9, 10, 11$.

The index, $q = 347$, is called the *rank* of the waveform. It designates the place of any particular waveform within an ordered list of all possible waveforms of a given class and a given order. The example in equation 110 is the 347th waveform in the list of all waveforms, which consist of four trigonometric factors (order 4) and which use the twelfth harmonic as the highest harmonic in any factor.

The method of ordering all these waveforms follows a block structure. We call a *block* an arrangement of 2^p trigonometric product waveforms where all waveforms belong to the same order and the same class, *and* all have exactly the same harmonic factors. The waveform of equation 110 belongs to the block :

$$(1,\ 3,\ 7,\ 12) \tag{111}$$

The ranking within a block follows the binary counting method but, contrary to the usual practice, the lowest ranking digit is on the left side and the highest ranking digit is on the right side. The sine is lower ranking than the cosine. The first waveform (lowest rank) is therefore the waveform with all sine factors. The last waveform (highest rank) is the waveform with all cosine factors. The block with the harmonic factors of the group in expression 111 above has 16 waveforms that must be listed in the following order :

$$\begin{aligned}
\#1 &: \sin x \,.\, \sin 3x \,.\, \sin 7x \,.\, \sin 12x \\
\#2 &: \cos x \,.\, \sin 3x \,.\, \sin 7x \,.\, \sin 12x \\
\#3 &: \sin x \,.\, \cos 3x \,.\, \sin 7x \,.\, \sin 12x \\
\#4 &: \cos x \,.\, \cos 3x \,.\, \sin 7x \,.\, \sin 12x \\
\#5 &: \sin x \,.\, \sin 3x \,.\, \cos 7x \,.\, \sin 12x \\
\#6 &: \cos x \,.\, \sin 3x \,.\, \cos 7x \,.\, \sin 12x \\
\#7 &: \sin x \,.\, \cos 3x \,.\, \cos 7x \,.\, \sin 12x \\
\#8 &: \cos x \,.\, \cos 3x \,.\, \cos 7x \,.\, \sin 12x \\
\#9 &: \sin x \,.\, \sin 3x \,.\, \sin 7x \,.\, \cos 12x \\
\#10 &: \cos x \,.\, \sin 3x \,.\, \sin 7x \,.\, \cos 12x \\
\#11 &: \sin x \,.\, \cos 3x \,.\, \sin 7x \,.\, \cos 12x
\end{aligned} \tag{112}$$

#12 : cos x . cos 3x . sin 7x . cos 12x

#13 : sin x . sin 3x . cos 7x . cos 12x

#14 : cos x . sin 3x . cos 7x . cos 12x

#15 : sin x . cos 3x . cos 7x . cos 12x

#16 : cos x . cos 3x . cos 7x . cos 12x

The total number of blocks in order p and class k can be found by computing the total number of different arrangements than can be made when filling k-1 positions with p-1 numbers. Notice that the k-th position is always filled (with the number k). In the example at hand there are twelve positions, eleven of which can be filled with three numbers. Marking the unfilled positions (trivial factors) with a dot, one arrives at the following first arrangements.

1 2 3 12

1 2 . 4 12

1 . 3 4 12

. 2 3 4 12

1 2 . . 5 12

1 . 3 . 5 12

. 2 3 . 5 12

1 . . 4 5 12

. 2 . 4 5 12

. . 3 4 5 12

1 2 . . . 6 12

etc.

until the last arrangement

. . . . 9 10 11 12

The total number of blocks is :

$$N_B = \frac{(k-1)!}{(p-1)!\,(k-p)!} \tag{113}$$

There are 165 blocks of waveforms of order 4 and class 12. Each block has 16 waveforms, making a total of 2640 waveforms.

It is easy to see that the total number of TPW (the highest rank q_m) in any particular subset of order p and class k is therefore :

$$q_m = 2^p \,.\, N_B = 2^p \,.\, \frac{(k-1)\,!}{(p-1)\,!\,(k-p)\,!} \tag{114}$$

When considering that $0\,! = 1$, one can readily compute the following Table 3 of all waveforms of all classes and orders up to $p = 4$, $k = 4$. We call the set of all waveforms of all classes and all orders up to and including $k = p = L$, the set of all waveforms of *level* L. Table 3 lists all waveforms of level 4. S stands for sine; C, for cosine. The column in which S or C is listed gives the harmonic factor. The waveform $f_{3,\,4,\,3}(x)$, listed as the third waveform in the group 11, is therefore defined as :

$$f_{3,\,4,\,3}(x) = \sin x \,.\, \cos 2x \,.\, \sin 4x$$

Table 3 shows further that group 11 contains all waveforms of order 3 and class 4. There are $q_m = 24$ waveforms, which are arranged in $N_B = 3$ blocks of $2^3 = 8$ waveforms each.

Table 3

All trigonometric product waveforms and blocks at level 4

Indices	Harmon. Factors			
p, k, q	1	2	3	4
1 $p=0;\ k=0;\ q_m=0;\ N_B=0$				
0, 0, 0	Trivial Case			
2 $p=1;\ k=1;\ q_m=1;\ N_B=1$				
0, 1, 1	$\sin\frac{\pi}{2} = 1$			
3 $p=1;\ k=1;\ q_m=2;\ N_B=1$				
1, 1, 1	S			
1, 1, 2	C			
4 $p=1;\ k=2;\ q_m=2;\ N_B=1$				
1, 2, 1		S		
1, 2, 2		C		
5 $p=1;\ k=3;\ q_m=2;\ N_B=1$				
1, 3, 1			S	
1, 3, 2			C	
6 $p=1;\ k=4;\ q_m=2;\ N_B=1$				
1, 4, 1				S
1, 4, 2				C
7 $p=2;\ k=2;\ q_m=4;\ N_B=1$				
2, 2, 1	S	S		
2, 2, 2	C	S		
2, 2, 3	S	C		
2, 2, 4	C	C		
8 $p=2;\ k=3;\ q_m=8;\ N_B=2$				
2, 3, 1	S		S	
2, 3, 2	C		S	
2, 3, 3	S		C	
2, 3, 4	C		C	
2, 3, 5		S	S	
2, 3, 6		C	S	
2, 3, 7		S	C	
2, 3, 8		C	C	
9 $p=2;\ k=4;\ q_m=12;\ N_B=3$				
2, 4, 1	S			S
2, 4, 2	C			S
2, 4, 3	S			C
2, 4, 4	C			C
2, 4, 5		S		S
2, 4, 6		C		S
2, 4, 7		S		C
2, 4, 8		C		C
2, 4, 9			S	S
2, 4, 10			C	S
2, 4, 11			S	C
2, 4, 12			C	C
10 $p=3;\ k=3;\ q_m=8;\ N_B=1$				
3, 3, 1	S	S	S	
3, 3, 2	C	S	S	
3, 3, 3	S	C	S	
3, 3, 4	C	C	S	
3, 3, 5	S	S	C	
3, 3, 6	C	S	C	
3, 3, 7	S	C	C	
3, 3, 8	C	C	C	
11 $p=3;\ k=4;\ q_m=24;\ N_B=3$				
3, 4, 1	S	S		S
3, 4, 2	C	S		S
3, 4, 3	S	C		S
3, 4, 4	C	C		S
3, 4, 5	S	S		C
3, 4, 6	C	S		C
3, 4, 7	S	C		C
3, 4, 8	C	C		C
3, 4, 9	S		S	S
3, 4, 10	C		S	S
3, 4, 11	S		C	S
3, 4, 12	C		C	S
3, 4, 13	S		S	C
3, 4, 14	C		S	C
3, 4, 15	S		C	C
3, 4, 16	C		C	C
3, 4, 17		S	S	S
3, 4, 18		C	S	S
3, 4, 19		S	C	S
3, 4, 20		C	C	S
3, 4, 21		S	S	C
3, 4, 22		C	S	C
3, 4, 23		S	C	C
3, 4, 24		C	C	C
12 $p=4;\ k=4;\ q_m=16;\ N_B=1$				
4, 4, 1	S	S	S	S
4, 4, 2	C	S	S	S
4, 4, 3	S	C	S	S
4, 4, 4	C	C	S	S
4, 4, 5	S	S	C	S
4, 4, 6	C	S	C	S
4, 4, 7	S	C	C	S
4, 4, 8	C	C	C	S
4, 4, 9	S	S	S	C
4, 4, 10	C	S	S	C
4, 4, 11	S	C	S	C
4, 4, 12	C	C	S	C
4, 4, 13	S	S	C	C
4, 4, 14	C	S	C	C
4, 4, 15	S	C	C	C
4, 4, 16	C	C	C	C

For computer evaluation it is important to identify each waveform by its order, class, and rank. While order and class will normally be restricted to numbers with not more than two decimal digits, one can see that the rank can assume extremely high numbers. The number of all TPWs at a certain level is :

$$N_{WL} = 3^L \tag{115}$$

If we take $L = 64$ as a reasonable upper limit for practical systems, we recognize that we have to deal with a fantastically large total number of TPWs :

$$N_{WL} = 3^{64} = 3.43362. \times 10^{30} \tag{116}$$

This family of more than 3.4 decillions of different wave forms is a strong asset for any system working with TPWs, but it forces one to employ a reasonable procedure for uniquely identifying any one of the waveforms.

To identify the order, class, and rank of any desired waveform, one should write all factors of the desired waveform in the following component notation :

$$(\alpha_1 \beta_1 \;\; ; \; \alpha_2 \beta_2 \;\; ; \; \ldots \alpha_i \beta_i \;\; ; \; \ldots \alpha_{p-1} \beta_{p-1} \;\; ; \; \alpha_p \beta_p) \tag{117}$$

The α_i with $i = 1, 2, \ldots, p$ are the harmonic factors arranged in increasing order : $\alpha_1 < \alpha_2 < \ldots \alpha_i \ldots < \alpha_p$. The β_i represent the sine function $\beta_i = S$ or the cosine function $\beta_i = C$.

As an example we shall show how to find order, class, and rank for the waveform :

$$f(x) = \sin 2x \,.\, \cos 4x \,.\, \sin 5x \,.\, \cos 12x \tag{118}$$

This TPW is symbolized by (2S ; 4C ; 5S ; 12C). (119)

The order is the number of factors, $p = 4$ in this case. The class is the highest harmonic factor, $k = 12$ in this case. The rank can be calculated with the help of the following formula (120) when placing the value $\beta_{p-m} = 0$ for β_{p-m}, $m = 0, 1, \ldots p - 1$, if the β_i term in 117 is S ; and the value $\beta_{p-m} = 1$, if the β_i term in 117 is C. In the example of equation 118 one has the following α and β constants :

$$\alpha_1 = 2 \,;\;\; \alpha_2 = 4 \,;\;\; \alpha_3 = 5 \,;\;\; \alpha_4 = 12$$

$$\beta_1 = 0 \,;\;\; \beta_2 = 1 \,;\;\; \beta_3 = 0 \,;\;\; \beta_4 = 1$$

These constants together with $p = 4$ and $k = 12$ have to be used in equation 120 to get the rank :

$$q = 2^p \left[\sum_{i=0}^{p-1} \frac{1}{(p-2-i)\,!} \sum_{j=0}^{p-1} \frac{(\alpha_{p-1-i} - 2 - j)\,!}{(\alpha_{p-1-i} - p - j + i)\,!} \right] + \left(\sum_{m=0}^{p-1} 2^{p-1-m} \beta_{p-m} \right) + 1 \tag{120}$$

In the special example of expression 119 this results in the following numerical values :

$$q = 2^4 \left[\frac{1}{1.2} \left(\frac{1.2.3}{1} + \frac{1.2}{0\,!} \right) + \frac{1}{1} \left(\frac{1.2}{1} + \frac{1}{0\,!} \right) + \frac{1}{0\,!} \left(\frac{0\,!}{0\,!} \right) \right] + 8.1 + 4.0 + 2.1 + 1.0 + 1$$
$$= 16\,[4 + 3 + 1] + 10 + 1 = 139$$

When applying equation 120 it is necessary to recall the following identities :

$$0\,! = 1 \,;\;\; 1\,! = 1 \,;\;\; \frac{1}{(-n)\,!} = 0 \tag{121}$$

The expression in the bracket of equation 120 gives the number of waveforms in all blocks with a lower rank than the block in which the desired waveform is situated. The rank of the block in which a waveform is situated is therefore :

$$R_B = \sum_{i=0}^{p-1} \frac{1}{(p-2-i)\,!} \sum_{j=0}^{p-1} \frac{(\alpha_{p-1-i} - 2 - j)\,!}{(\alpha_{p-1-i} - p - j + i)\,!} + 1 \tag{122}$$

For example, the waveform sin 2x . sin 3x . cos 4x (2S ; 3S ; 4C) of table 3 is situated in order $p = 3$, class $k = 4$, and block number :

$$R_B = \frac{1}{1\,!} \left[\frac{(\alpha_2 - 2)\,!}{(\alpha_2 - 3 - 0 + 0)\,!} \right] + \frac{1}{0\,!} \left[\frac{(\alpha_1 - 2)\,!}{(\alpha_1 - 3 - 0 + 1)\,!} \right] + 1$$
$$= 1 + 1 + 1 = 3$$

The rank of the selected waveform is :

$$q_m = 8 \,.\, [2] + (2^2 \,.\, 1 + 2 \,.\, 0 + 1 \,.\, 0) + 1 = 21$$

With the help of the above specifications of order, class, rank, and, when applicable, of level, any TPW can be uniquely identified. It may be one of decillions of different waveforms.

O2. *The Signal Base of Trigonometric Product Waveforms*

The whole Part IV of this paper is concerned with alphabets of bandlimited transmission signals. In subsection M4 we explained the fundamentals of bandlimited waveforms and we showed the importance of the TB product or, rather, the importance of the signal base per dimension (β_c).

Before concentrating on the problem of the signal base of TPW sets, it may be advisable to show a number of typical TPWs and their energy density spectra. Figure 31 shows typical waveform pictures for a number of examples from first order to fourth order and from first class to eight class.

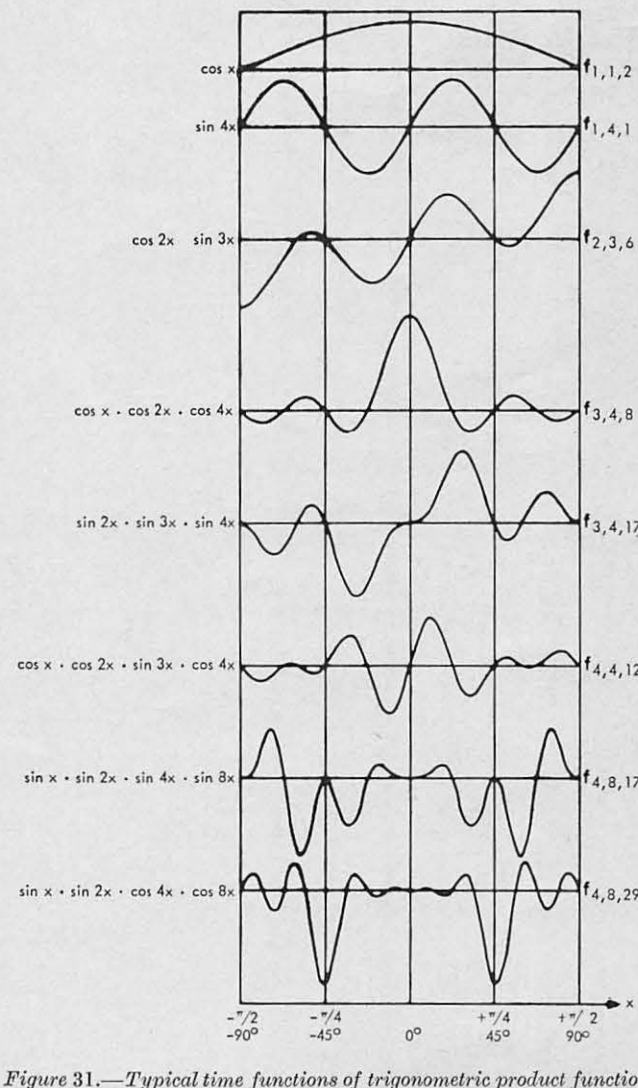


Figure 31.—Typical time functions of trigonometric product functions of various orders and classes.

The first two waveforms cos x and sin 4 x are both in the first order. Notice that a waveform of first order and first class comprises only one half oscillation of the fundamental sine or cosine wave. Thus we recognize that the trigonometric waveform alphabets discussed in subsection N3 are only a subset of all TPWs of order one. In particular they are the subset consisting of all waveforms contained in the set of TPWs of order one and of all even classes. The second example in fig. 31 (sin 4x) is one waveform of this subset.

Further inspection of fig. 31 shows that all the TPWs in this example are either even or odd functions. This is a fundamental characteristic of all TPWs. Evidently, it is always possible to find a quadrature pair of TPWs, where the one partner of the pair is the even waveform and the other partner is the odd waveform, permitting any linear combination of the two components to synthesize any unsymmetrical waveform of that special variety. Figure 31 shows that any waveform with an odd number of sine factors results in an odd TPW. Any waveform with an even number of sine factors results in an even TPW, no matter what harmonic factors are involved in the individual factors. This is easy to prove when considering that all sine factors are zero for $x = 0$ but have a positive derivative at $x = 0$, while all cosine factors are $+1$ at $x = 0$ with a zero derivative.

Figure 31 shows another interesting characteristic of TPWs. Waveforms with all cosine terms form a positive pulse at $x = 0$, while waveforms with all sine terms have an extended valley of $f(x) = 0$ near $x = 0$. Figure 31 also shows that only one of the eight waveforms selected as typical examples has a non-zero value at the boundaries of the definition interval. Indeed, it is easy to prove that only waveforms with all sine terms exclusively of odd harmonics and with all cosine terms exclusively of even harmonics can produce a non-zero value at the limits of the interval. Whenever this is the case, the non-zero value must be the peak value.

Though many of such fundamental characteristics of TPWs are evident from an inspection of waveform pictures, it is deplorable that no all-inclusive theory of TPWs has yet been developed. We are still in a kind of mathematically experimental evaluation of this new family of waveforms.

Figure 32 demonstrates the next logical step in the exploration of TPWs ; the computation of their energy density spectra. Again we have assembled a rather arbitrary selection of TPWs from order 1 to 6 and from class 3 to 16. The energy density spectra are plotted to the right of each waveform on a normalized scale over $\eta = f/f_o$ with $f_o = x/2\pi t$. This means that $\eta = 1$ corresponds to the fundamental frequency and that each waveform is defined over an interval from $-90°$ $(-\pi/2)$ to $+90°$ $(+\pi/2)$ of the angle of this fundamental frequency (f_o). The first waveform in fig. 32 (sin 6x) shows this very clearly by displaying a very definite maximum of energy density at $\eta = 6$.

Comparing all energy density spectra of fig. 32b with the corresponding waveform pictures of fig. 32a reveals clearly the fact that all waveforms are essentially bandlimited to a normalized frequency

$$\eta_L = 2 + \sum_{i=1}^{p} \alpha_i \quad , \tag{123}$$

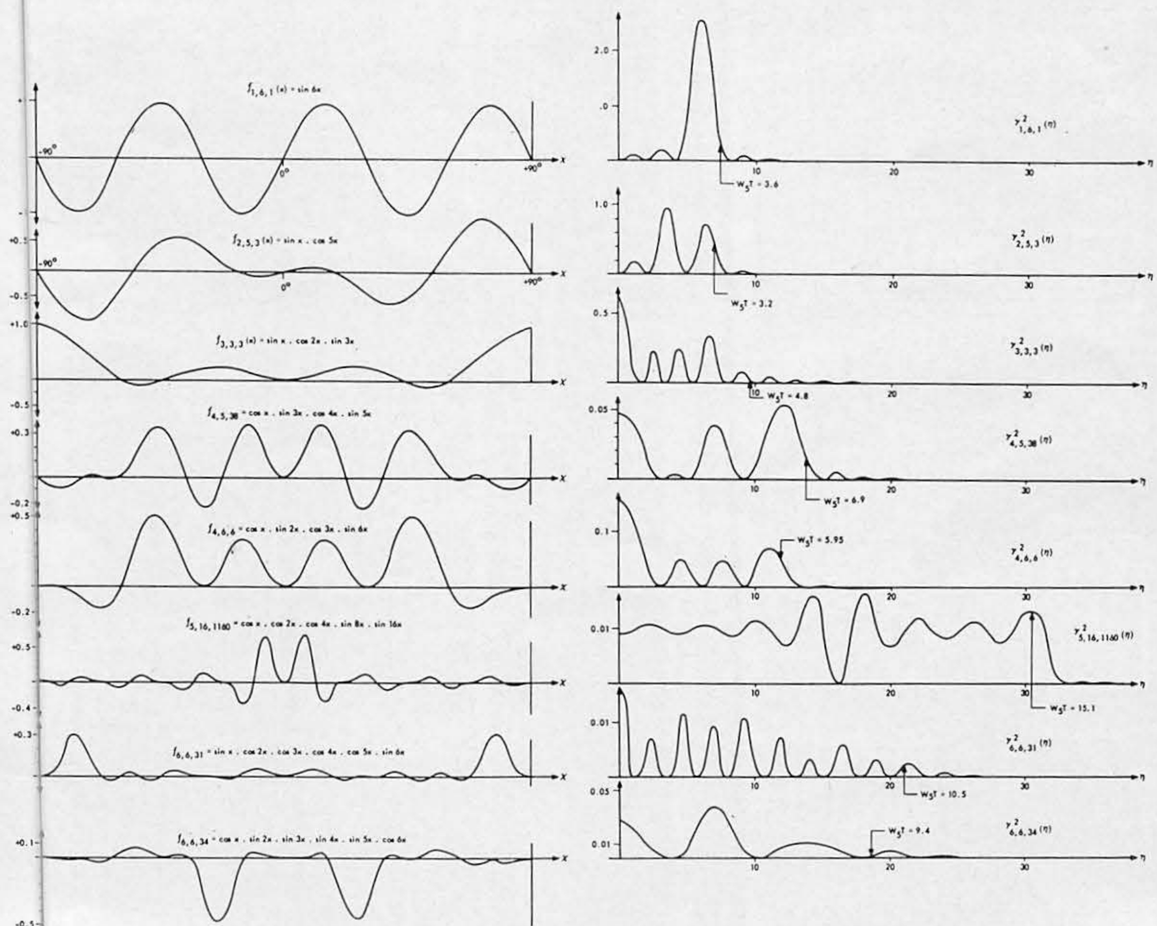


Figure 32.—Comparison of time functions and energy density spectra of a selection of TPWs.

where the α_i are the harmonic factors participating in the definition of any waveform. Take, for example, the first waveform with $\alpha_1 = 6$ ($\eta_L = 8$) or the second waveform with $\alpha_1 = 1$, $\alpha_2 = 5$; ($\eta_L = 8$), or the last waveform in fig. 32a with $\alpha_1 = 1$; $\alpha_2 = 2$; $\alpha_3 = 3$; $\alpha_4 = 4$; $\alpha_5 = 5$; $\alpha_6 = 6$ ($\eta_L = 23$). In each case we recognize a zero of the spectral density at η_L and we see that, for $\eta = \eta_L$, a zero occurs at each value for $\eta = \eta_L + 2n$ with n = 0, 1, 2, ... This means that the essential energy of any waveform stays within a frequency band from zero to η_L (in normalized frequencies) and that the energy density for frequencies $\eta > \eta_L$ is relatively small. Evidently, waveforms with discontinuities at the end of the time interval show a larger percentage of energy at frequencies $\eta > \eta_L$. Yet even these waveforms follow the rules established above.

This leaves us with the problem of exactly defining the TB product (β-value) for any given waveform and for any exactly defined percentage of waveform energy outside the TB product. Figure 33 shows how this can be done for various TB values. As TPWs are exactly defined in the time domain, the truncation here is made only in the frequency domain. A waveform truncated in the frequency domain will produce " tails " outside the definition interval in the time domain. No detailed investigation of this interdependence between frequency and time domain has been made yet for TPWs, although it is recognized that the results with PSWs (subsection N4) will be indirectly applicable for any such study. Figure 33 shows that this particular waveform (1S, 2S, 4S) has a TB product of 4.00 when about 2.6 percent of the energy can be tolerated outside the normalized bandlimit (η = 8.0 ; TB = 4.00). This sample waveform is one of a set of eight orthogonal waveforms which will be discussed in the next subsection.

Computer programs are available to plot any TPW up to rather high orders, classes, and ranks. The computer can be instructed to plot individual waveforms, blocks of waveforms, or whole orders and/or classes. The precision of plotting can be adjusted from a highest precision from a minute to minute to a quick-look plot from degree to degree. When plotting the waveforms from $-90°$ ($-\pi/2$) to $+90°$ ($+\pi/2$) with the finest plotting, the printout requires 180 sheets per waveform and with the coarsest plotting it needs only three sheets. The exact values of the waveform for each time increment can be printed out in six to eighteen digit numbers. The programs have been written for IBM 360/75 computer.

In addition to the program for waveform plotting, programs for the calculation of amplitude spectra and of energy density spectra are available. No systematic evaluation of the signal base for other than the classical orthogonal sets (to be discussed next) has been made up to this time.

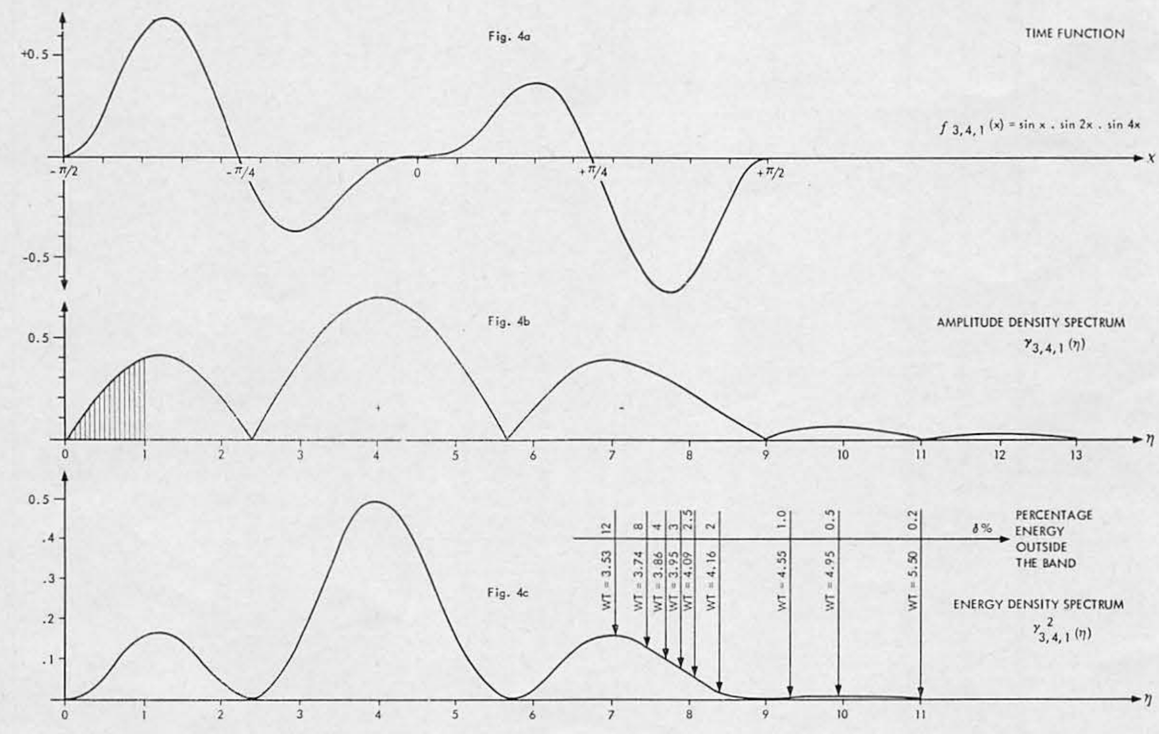
O3. *Orthogonal Sets of Trigonometric Product Waveforms*

Evidently the first goal of the investigation of trigonometric product waveforms was the search for orthogonal sets of bandlimited waveforms. Indeed this search was in the background of the author's mind when attempting to create better bandlimited waveforms by using a multiplicative approach to avoid discontinuities in the waveforms.

The first orthogonal set to be discovered, and for some time believed to be the only orthogonal set, was the " *classical orthogonal set* ". It is called classical because it has a most regular structure by making each harmonic factor α_i a power of two and by arranging all these powers of two in ascending order. A classical orthogonal block therefore follows the expression :

$$(1, 2, 4, 8, \ldots, 2^{p-1}) \tag{124}$$

Figure 34 (Filipowsky 67a) shows the eight waveforms and their energy density spectra of the classical orthogonal set of order 3. One can see that only one out of the 2^p waveforms of a classical orthogonal set has a discontinuity at the end of the interval. It has been stressed in subsection



WAVEFORM and SPECTRA of $f_{3,4,1}$ = sin x . sin 2x . sin 4x

Figure 33.—Definition of the TB values (signal bases) for a typical TPW.

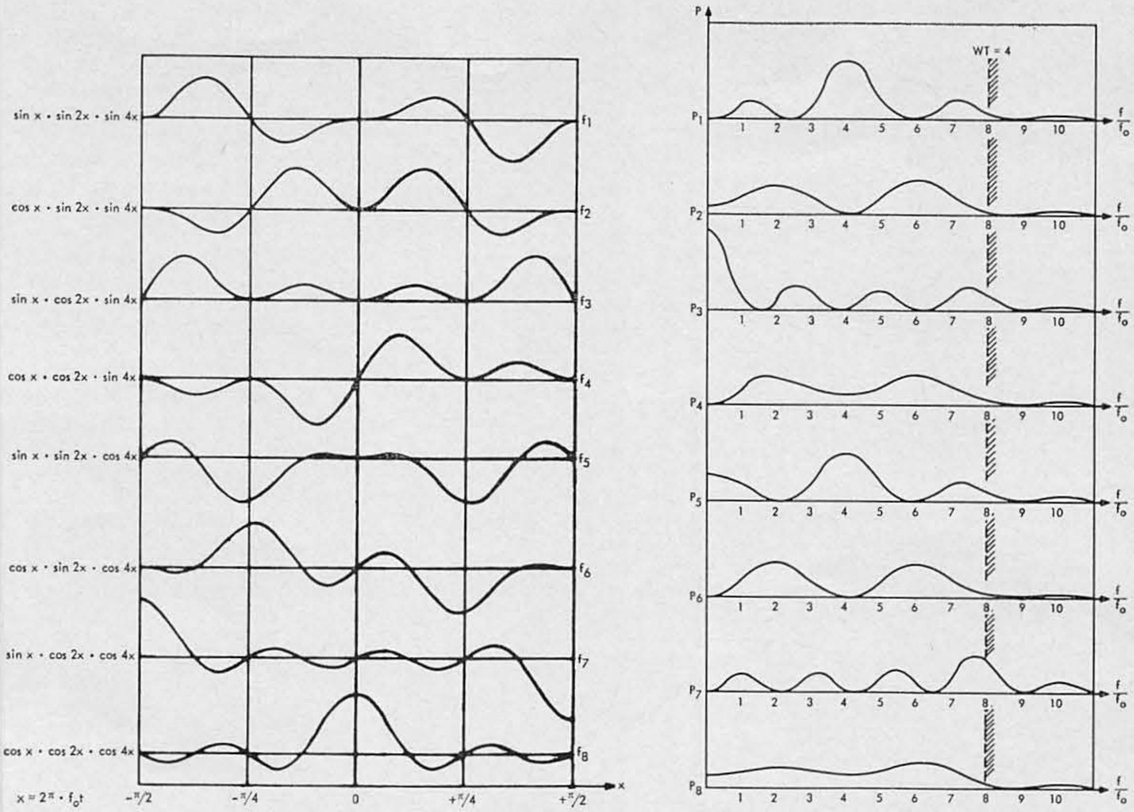


Figure 34.—The time functions and the spectra of the classical orthogonal set of TPWs of order 3.

O2 that this criterion for a discontinuity applies quite generally to all TPWs, not only to orthogonal sets. Indeed, it can be shown that only one combination of sines and cosines in each block of waveforms can fulfill this criterion. To find in any block this one waveform with the discontinuities, one must use the sine function for all odd harmonics and the cosine function for all even harmonics. Therefore, in all classical orthogonal sets, the waveform with the structure (1S, 2C, 4C, 8C, . . . , 2^{p-1}C) must be the one with the discontinuity. According to our ranking principle this is always the next to last waveform in a classical orthogonal set. This singular occurrence of a waveform with discontinuities is one of the most attractive features of TPWs. The larger the order of the waveform, the smaller is the percentage of waveforms with discontinuities. Evidently, the TPWs of the first order have 50 percent of all waveforms with discontinuities. This is still the case when only classes of even harmonics are used for first order waveforms, as proposed for the trigonometric sets of section N3. One may get a subset of first order waveforms that is free of discontinuities by using only sine functions for the waveforms with an even harmonic factor and only cosine functions for the waveforms with an odd harmonic factor, thus omitting 50 percent of all first order TPWs. Still even such subsets will have a strong spill-over in the frequency domain due to the discontinuities in all derivatives of such subsets. TPWs of higher order are definitely preferable.

To avoid any harmful effect of that one waveform with a discontinuity in any block of waveforms of higher order, orthogonal or nonorthogonal, the suggestion has been made (Filipowsky 68b) to use this waveform for synchronization purposes. If it is used with alternate polarity in alternate intervals and if it is used with constant amplitudes, the spectrum of this waveform degenerates to a line spectrum with the highest frequency component at $\eta = 7$ (in the example of fig. 34). If the set is an orthogonal set and if exactly clocked resampling is applied, the synchronization waveform can be applied continuously without interfering with the other waveforms that may be used for message transmission.

There is one other characteristic waveform in fig. 34 and that is the last one of the set. It has cosine factors only and it resembles a bell-shaped pulse or a sine x over x pulse at the centre of each transmission interval. The energy density spectrum of such a pulse has a nearly uniform distribution over the available frequency band. Accordingly, the suggestion has been made (Filipowsky 68b) to avoid using this waveform for message transmission but rather use it in the receiver correlator as a "*noise window*". This means that the noisy composite signal in the receiver will be correlated with this waveform as it will be correlated with all the other waveforms. Since this waveform never will be transmitted (no component of the message vector will be in the dimension of this waveform), the correlator output corresponding to this waveform will be due only to external disturbances (noise). Thus the receiver can "look-through" the message signals to see the noise. Because of the uniform frequency spectrum of this "window waveform", it is likely that any random noise components will be traced. Noise components in the exact dimensions of the other waveforms cannot be noticed. Obviously, these components are the ones causing errors; evidently, the noise window cannot compute the exact position of such noise vectors. Yet, the larger the dimensionality of the set, the smaller will be the probability that a random noise vector will have zero component value in the dimension of the noise window and high component values in any other dimension. Indeed, for many practical systems, small band disturbances (cross talk from other carriers, CW jamming, etc.) will be the principal noise source. Any such disturbance must have a significant component in the dimension of the noise window. Averaging the noise window output over many intervals

can give a good indication of any quasistationary random disturbance. Evasive measures can then be taken in any adaptive communications system.

Tables 4 and 5 give the TB products of the classical orthogonal sets of order 3 and order 4 respectively. In these tables the waveforms are listed in the order of their signal base (TB), with the waveform with the smallest signal base coming first. The waveform with the discontinuity has the largest signal base and therefore comes last in the tables. It can be estimated from the values of table 5 that 15 waveforms, when permitting 12 percent of their energy outside the TB value, would have an average signal

Table 4

TB products for the classical orthogonal set of TPWs of order 3

Typical waveform equation: sin x· sin 2x· sin 4x						
Number	% of waveform energy outside TB					
	2%	4%	6%	8%	10%	15%
$f_{3,4,2}$	3.85	3.68	3.57	3.50	3.43	3.28
$f_{3,4,6}$	3.85	3.68	3.57	3.50	3.43	3.28
$f_{3,4,4}$	3.95	3.72	3.60	3.52	3.45	3.34
$f_{3,4,8}$	3.95	3.72	3.60	3.52	3.45	3.34
$f_{3,4,1}$	4.16	3.86	3.82	3.74	3.64	3.42
$f_{3,4,5}$	4.18	3.92	3.86	3.78	3.70	3.48
$f_{3,4,3}$	4.82	4.08	3.96	3.88	3.78	3.62
$f_{3,4,7}$	8.40	7.40	6.80	6.70	5.80	5.20

Table 5

TB products for the classical orthogonal set of TPWs of order 4

f#	Harmonics				Percentage δ of energy outside TB						
	1	2	3	4	0.1%	0.5%	1%	2%	3%	5%	10%
$f_{4,8,22}$	C	S	C	C	8.17	7.96	7.85	7.69	7.61	7.45	7.19
$f_{4,8,30}$	C	S	C	C	8.18	7.98	7.85	7.67	7.60	7.47	7.20
$f_{4,8,18}$	C	S	S	S	8.18	7.92	7.83	7.76	7.58	7.42	7.18
$f_{4,8,26}$	C	S	S	C	8.18	7.96	7.83	7.71	7.60	7.42	7.19
$f_{4,8,20}$	C	C	S	S	8.32	8.03	7.91	7.74	7.63	7.49	7.18
$f_{4,8,28}$	C	C	S	C	8.32	8.03	7.91	7.74	7.69	7.36	7.18
$f_{4,8,24}$	C	C	C	S	8.50	8.08	7.95	7.76	7.67	7.50	7.18
$f_{4,8,32}$	C	C	C	C	8.82	8.10	7.97	7.81	7.68	7.51	7.21
$f_{4,8,17}$	S	S	S	S	9.00	8.10	8.00	7.86	7.80	7.61	6.78
$f_{4,8,25}$	S	S	S	C	9.04	8.18	8.04	7.89	7.77	7.63	6.74
$f_{4,8,21}$	S	S	C	S	9.25	8.22	8.11	7.93	7.84	7.66	7.10
$f_{4,8,29}$	S	S	C	C	10.25	8.50	8.15	7.97	7.86	7.70	7.23
$f_{4,8,19}$	S	C	S	S	10.91	9.04	8.31	8.05	7.96	7.81	7.42
$f_{4,8,27}$	S	C	S	C	14.00	10.00	9.00	8.19	8.08	7.91	7.70
$f_{4,8,23}$	S	C	C	S	18.50	12.52	10.75	9.21	8.81	8.12	7.78
$f_{4,8,31}$	S	C	C	C	29.50	25.00	22.50	19.00	17.20	14.52	11.10

base of about 7.04. This means that the number of waveforms in this orthogonal set (15) is about 2.13 times the signal base ; i.e., the theoretical value of 2.4 times the signal base is being approached. No computations have been made for the signal bases of sets of higher order (beyond p = 4), but a comparison of table 4 with table 3 does permit the expectation that sets of seventh or eighth order (128 or 256 waveforms) will come very close to the theoretical minimum signal base. More important may be the performance under extremely tight band-limitation. Table 5 shows that 12 of the 16 waveforms have a signal base of 10.25 or less when only 0.1 percent of the energy can be permitted outside the band and time limits. It will be interesting to see how these figures improve still further when going to sets of still higher order.

This author was rather surprised to discover that there exist many more sets of orthogonal TPW than merely the classical orthogonal sets. Yet the indications are that the classical sets are the best bandlimited sets. Naturally, it was important to find an algorithm that can indicate from the harmonic factors of a block if this block will represent an orthogonal set or not. Such an *orthogonal algorithm* has recently been found. One simply inspects all partial sums that can be formed from all the harmonic factors of a block. If any two partial sums (which must not contain the same harmonic factor twice) are equal, the block is nonorthogonal. If no such sums are equal, the block is orthogonal. Single harmonic factors count as a partial sum. A few examples will show this procedure. We may take some of the blocks of order 4 and class 12 as they are listed in subsection O1 :

(1, 2, 3, 12)	nonorthogonal	$1 + 2 = 3$
(1, 2, 4, 12)	orthogonal	
(1, 3, 4, 12)	nonorthogonal	$1 + 3 = 4$
(1, 3, 7, 12)	orthogonal	
(3, 4, 5, 12)	nonorthogonal	$3 + 4 + 5 = 12$
.		
.		
.		
(9, 10, 11, 12)	nonorthogonal	$9 + 12 = 10 + 11$

Combining this investigation with equation 123 for bandlimitation indicates that the clasical orthogonal set is usually the set of the lowest class in a given order. As the class is increased, more and more orthogonal sets can be found in the same order, until the class is high enough to yield a classical orthogonal set of the next higher order. To demonstrate this behaviour, we start with order 4. The classical set is (1, 2, 4, 8) with $\eta_L = 17$; there are 8 other orthogonal sets in this order and class. They are tabulated below together with η_L, their normalized bandlimitation (see equation 123).

(2, 3, 4, 8)	$\eta_L = 19$	(4, 5, 6, 8)	$\eta_L = 25$
(2, 4, 5, 8)	$\eta_L = 21$	(2, 4, 7, 8)	$\eta_L = 23$
(1, 4, 6, 8)	$\eta_L = 21$	(3, 6, 7, 8)	$\eta_L = 26$
(3, 4, 6, 8)	$\eta_L = 23$	(4, 6, 7, 8)	$\eta_L = 27$

However, there is one orthogonal set of order 4 and a class lower than 8. This is (3, 5, 6, 7) with $\eta_L = 23$. Continuing this search for orthogonal sets in order 4 one will find eleven orthogonal sets for class 9, thirty-four sets for class 10, forty-five sets for class 11, seventy-seven orthogonal sets for class 12, and so forth. All these sets of classes higher than 2^{p-1} require wider bandwidth than the classical set, which is of class 2^{p-1}.

Other interesting properties of orthogonal sets and also of nonorthogonal sets are still under investigation at the time of this writing. Autocorrelation programs and cross-correlation programs have been written for the IBM 360/75 computer. Cross-correlation matrices can be easily plotted and noise (in the form of random samples of a given distribution) can be added. Preliminary results of this continuing investigation seem to indicate that at least the classical orthogonal sets seem to be autocorrelation invariant. Many similar conjectures can be verified experimentally on the computer, but rigorous mathematical proofs for many of the outstanding characteristics of TPW are still missing.

O4. *Future Possibilities for Trigonometric Product Waveforms*

Much research is still required before claims can be made that we really know all the characteristics of TPWs. First of all, it will be necessary to establish the connection to the still wider class of trigonometric polynomials. Evidently each TPW of order p can be expanded into a trigonometric polynomial of 2^{p-1} terms. Second, it will be beneficial to investigate TPWs in the complex domain and establish their relationship to orthogonal exponentials. Third, there is not only the large family of nonorthogonal blocks, but also the cross-correlation between different orthogonal sets that await further investigation.

There is no doubt, however, that the characteristics of TPWs that have been uncovered so far will assure the continued interest in these waveforms. One may remember that sine and cosine functions are the backbone of all communications and they temselves are derived from the wheel, the greatest invention of the human race.

Conclusions

This comprehensive review of the fundamentals of nonbinary information transmission has shown :

1. Nonbinary systems are theoretically superior to binary systems.
2. Multidimensional nonbinary systems are essential for significant progress in this area.
3. Multidimensional codebooks have a long history of research and many of the results of error-correcting codes can be applied to systems with message decoding.
4. The research for bandlimited waveforms has lagged behind the codebook research but it is now catching up.
5. Only a few practical nonbinary systems are in operation at this time, but many experimental systems are under investigation.
6. When attempting to close the margin of 20 to 30 decibels of utility that still separates the experimental systems from their mathematical models, designers will require large scale integration technology and computer-like system organization.

Acknowledgments

This survey has been prepared by the author in the years 1967 and 1968 with the encouragement and the support by the management of the Huntsville, Alabama, facility of the Space Systems Center of International Business Machine Corporation's Federal Systems Division.

The author gratefully acknowledges the invaluable assistance of the facility's librarian, Miss Dorothy Johnson, of the library staff and of the IBM secretaries, Miss Ivette Sudmalis and Mrs. Becky Head.

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APPENDIX A

Appendix A

List of Abbreviations

A/D	Analog-to-digital (conversion)
ADEM	Adaptively data equalized modem
AFC	Automatic frequency control (modem)
AFSK	Audio frequency shift keying
AGC	Automatic gain control (= AVC, automatic volume control)
AM	Amplitude modulation
APR	Automatic phase reference (class)
ARQ	Automatic request for repetition (Morse code symbol)
ASK	Amplitude shift keying (on-off keying)
BC	Bose-Chaudhuri (code)
BCH	Bose-Chaudhuri-Hoghem (code)
BDD	Bounded distance decoding
BFC	Binary fading channel
bps	Bits per second
BTL	Bell Telephone Laboratories
CCTR	Carrier coherent telemetry rate (subsystem)
CW	Continuous wave (unmodulated carrier)
db	Decibel (logarithmic unit)
dc	Direct current (constant value component)
DC-MPSK	Differentially coherent-multiphase shift keying (system)
DCPSK	Differentially coherent phase shift keying (system)
DDT	Digital data transceiver
DELTIC	Delay line time compressor
DNRZ	Differential non-return to zero
DSB-SC	Double sideband-suppressed carrier (modulation)
DSIF	Deep space instrumentation facility
exp	Mathematical symbol for the exponential function
FD	Frequency division (in multiplex systems)
FM	Frequency modulation
FMDFB	FM demodulator with feedback (also FMFB)
FSK	Frequency shift keying
HF	High frequency
HOA	Higher order alphabet (nonbinary)
IF	Intermediate frequency (also i.f.)
ISIC	Intersymbol interference corrector
JPL	Jet Propulsion Laboratory
LES	Lincoln experimental satellite
LET	Lincoln experimental terminal
LSI	Large scale integration

MASK	Multiamplitude shift keying = multilevel system
MDD	Minimum distance decoding
MF	Medium frequencies (300 kHz to 3 MHz)
MFSK	Multifrequency shift keying
MIT	Massachusetts Institute of Technology
MLD	Maximum likelihood decoder (or decider)
MODEM	Modulator-demodulator (combination)
MPSK	Multiphase shift keying
MTSK	Multitime shift keying
MUX	Multiplex (operation, system, equipment, etc.)
NCFSK	Non-coherent frequency shift keying (system)
NRZ	Non-return-to-zero
PAM	Pulse amplitude modulation
PAR	Peak-power to average-power ratio
PCM	Pulse code modulation
PhM	Phase modulation (analog)
PLL	Phase lock loop
PLLFB	Phase lock loop with feedback
PMCM	Pulse Morse code modulation
PN	Pseudo-noise
PR	Pseudo-random
P/S	Parallel-to-serial (converter)
PSI	Pseudo-sample insertion (technique)
PSK	Phase shift keying (system)
PST	Paired selective ternary (transmission)
PSW	Prolate spherical wave functions
PWS	Predicted wave signalling (Kineplex)
QFT	Quantized frequency transmission
QPPM	Quantized pulse position modulation (system)
RC	Resistor-capacitor (circuits)
RCA	Radio Corporation of America
RM	Reed-Müller (codes)
RMS	Root-mean-square (value)
RPT	Relative phase telegraphy
RZ	Return-to-zero
SD	Synchronous demodulator
SECO	Self-regulating error correcting coder-decoder
SNR	Signal-to-noise (ratio)
S/P	Serial-to-parallel (conversion)
SPR	Separate phase reference (class)
SR	Shift register
SSB	Single sideband (modulation)
TASI	Time assignment speech interpolation (system)
TB	Time bandwidth (product)
TPC	Time polarity control
TPW	Trigonometric product waveforms
USAF	United States Air Force
VCO	Voltage controlled oscillator
VIF	Vestigial interpolation filter
VOCODER	Voice coder and decoder
VSB	Vestigial sideband (modulation)
WJ	Wozencraft and Jacobs (65), (reference to their book, " Principles of Communications Engineering ")
XMTR	Transmitter (also Xmtr)

APPENDIX B

LIST OF SYMBOLS

Symbol	Quantity	Unit	MKS Dimension
A	SNR parameter used by Shannon (59). ($A = \sqrt{\sigma}$)	Number	[1]
a_{ij}	Coordinates in orthonormal signal space.	Number	[1]
B	Bandwidth (in transmission channels, equal to B_{eff} unless stated differently).	Hertz	[s^{-1}]
B_{eff}	Effective bandwidth	Hertz	[s^{-1}]
B_{in}	Effective bandwidth of the information-carrying signals at the input to the transmitter.	Hertz	[s^{-1}]
B_{out}	Effective bandwidth of the information-carrying signals at the output from the receiver.	Hertz	[s^{-1}]
B_S	Effective bandwidth of one subchannel in a MFSK system.	Hertz	[s^{-1}]
B_{tr}	Effective bandwidth of the transmission channel (if specifically designated).	Hertz	[s^{-1}]
C	Shannon's channel capacity.	Bit per second	[1]
C_{SH}	Shannon's channel capacity in nits (natural logarithmic units) per dimension (degree of freedom).	Nit per dimension	[1]
$C(l, D)$	The " constant " of a particular exponential bound of the error probability of coded multidimensional systems.	Number	[1]
D_B or D	Bit density (= transmission rate over bandwidth).	Number or decibel	[1]
D_1	Bit density of Shannon's ideal system.	Number	[1]
db	Symbol for decibel indicating that ten times logarithm of base ten has to be used of the magnitude to which db is placed as superscript.	Symbol	—
d_{ij}	Distance between the i-th and j-th code word.	Number	[1]
E	Average signal energy (average over all members of a set of signals).	Watt	[VAs]
E_B	Average signal energy per bit of output information.	Watt second per bit	[VAs]
E_D	Average signal energy of one digit.	Watt second per digit	[VAs]
$E_L(l, D)$	Reliability function of an exponential bound.	Number	[1]
E_s	Energy of a particular signal f_s.	Watt second	[VAs]
$\bar{e}$	Error ratio ; the bar indicates that error counts need to be averaged over long intervals.	Number	[1]
$\bar{e}_B$	Bit error ratio.	Number	[1]
$\bar{e}_C$	Character error ratio.	Number	[1]
$\bar{e}_D$	Digit error ratio.	Number	[1]
e_{EB}	Equivalent bit error ratio.	Number	[1]
$\bar{e}_W$	Word error ratio (average probability of a wrong decision when receiving one word out of a codebook of M words).	Number	[1]

Symbol	Quantity	Unit	MKS Dimension
f_0	Fundamental frequency of a TPW.	Hertz	$[s^{-1}]$
K_{BO}	Boltzman's constant ($1.38044.10^{-23}$)	Watt second per degree Kelvin	$[VAs\ {}^\circ K^{-1}]$
k	Number of digits in a sequence of digits. Also: class of a trigonometric product waveform.	Number	[1]
L	The largest possible number of orthonormal functions that can exist in a certain TB product under specified restraints. Also: level of trigonometric product waveforms.	Number	[1]
L′	A numerical magnitude used in Part III.	Number	[1]
l or l	Number of elements in a sequence of elements (in a digit or in a word).	Number	[1]
M	Number of rows (code words) in the codebook matrix (number of different sequences of digits available for transmission).	Number	[1]
m	Number of digits (states or levels in a nonbinary signal alphabet).	Number	[1]
m_i	Number of elements in the i-th group of numbers in permutation modulation.	Number	[1]
m_o	Number of dimensions of a multidimensional signal.	Number	[1]
m_C	Number of characters in a nonbinary output alphabet.	Number	[1]
N	Average noise power in a given bandwidth.	Watt	[VA]
N_o	Noise power density of white noise.	Watt second	[VAs]
NF	Noise figure (in decibels) or noise factor (in numbers).	Number or decibel	[1]
n	Number of bits carried by one digit or by one sequence of digits (see R_{NH}).	Number	[1]
n(t)	Noise waveform, random function.	*Neutralized	$[V^{1/2}A^{1/2}]$
P	"Power" of a code word (see fig. 16).	Number	[1]
P_c	Probability of correct decision.	Number	[1]
p	Order of a trigonometric product waveform.	—	—
q	Number of waveforms in the waveform library. Also: rank of a TPW.	Number	[1]
q_0	Number of orthonormal functions in a set of such functions.	Number	[1]
Q(x)	A special form of the probability integral (see fig. 8b).	Number	[1]
R	Transmission rate	Bit per second or other units.	$[s^{-1}]$
$\bar{R}$	Average transmission rate (for time-variable systems).	Bit per second or other units	$[s^{-1}]$
R_B	Transmission rate.	Bit per second	$[s^{-1}]$
R_0	Binary exponential bound parameter as used by Wozencraft and Jacobs (65).	Number	[1]
R_0*	Nonbinary exponential bound parameter as used by Wozencraft and Jacobs (65).	Number	[1]
R_{NH}	Nyquist and Hartley's information content of a nonbinary word (also, see n).	Bit per word	[1]
R_{SH}	Shannon's information rate in nits per dimension.	Nits per dimension	[1]
r	Radius of a sphere in signal space.	*Neutralized	$[V^{1/2}A^{1/2}]$
S	Average signal power taken over many message intervals (assuming stationary signal and message statistics).	Watt	[VA]
$\hat{S}$	Peak signal power during any signal.	Watt	[VA]
S_i	Average signal power of the i-th signal waveform of an alphabet of q waveforms or of the i-th message element of a message sequence of l elements	Watt	[VA]
s_i (t)	I-th signal function (waveform).	*Neutralized	$[V^{1/2}A^{1/2}]$
T_B	Bit period, the reciprocal value of R.	Second	[s]
T_C	Duration of one time slot reserved for one character of the waveform library.	Second	[s]
T_D	Duration of one digital interval (usually identical with T_C).	Second	[s]
T_M	Duration of one message interval.	Second	[s]
$\bar{T}_{op}$	Average operational noise temperature.	Degree Kelvin	[°K]
T_W	Duration of one transmission word consisting of l time slots of length T_C each.	Second	[s]
u	Utility (= bit density over power contrast).	Number or decibel	[1]
u_1	Utility of Shannon's ideal system.	Number	[1]
$\bar{w}_i$	Word vector in l dimensional codebook space.	Vector	[1]
$\bar{w}_{LR}$	Transmission vector (word) in linear real coding (LR).	Vector	[1]
$\bar{a}$	Input vector with real numbers a_i as coefficients.	Vector	[1]
a_i	i-th harmonic factor defining a TPW.	Number	[1]
β	Signal base (TB product).	Number	[1]
β_B	Signal base for one bit of transmitted information.	Number	[1]
β_C	Signal base of one character of the waveform library or of one chip in binary sequences.	Number	[1]
β_D	Signal base for one digit in a nonbinary system.	Number	[1]
β_M	Signal base for one message interval.	Number	[1]
β_S	Signal base for one waveform in one subchannel of a MFSK system.	Number	[1]
β_W	Signal base for one transmission word (of a codebook).	Number	[1]
β_i'	Elements in examples of permutation modulation (see fig. 18).	Number	[1]
Δ	A numerical term used in Shannon's theorem of othornormal expansion.	Number	[1]
$\Delta(\sigma)$	A constant depending only on the SNR and used for the design of approximate utility curves.	Number	[1]
δ	Factor relating the bit density of a mathematical model to D_1 by the equation $D = \delta D_1$.	Number	[1]

**Neutralized Unit*: Time functions may be voltages or currents. To avoid specifying two physical magnitudes, many authors prefer to apply neutralized units with the dimension $(\text{watt})^{1/2}$. Any time function, when expressed in neutralized units, will supply to a one-ohm resistor a power equal to the square of the time function value, whether representing a voltage or a current.

Symbol	Quantity	Unit	MKS Dimension
ϵ	Energy contrast (E over N_o).	Number or decibel	[1]
ϵ_B	Energy contrast for one bit of transmitted information.	Number or decibel	[1]
$\bar{\epsilon}_C$	Average energy contrast for one character of the waveform library.	Number or decibel	[1]
ϵ_D	Energy contrast for one digit in a nonbinary system.	Number or decibel	[1]
ϵ_Δ	Error of approximation when using an expansion into orthonormal components.	Number	[1]
η	SNR difference between a mathematical model and Shannon's ideal system. Also: normalized frequency in the theory of TPW.	Number or decibel	[1]
η_L	The normalized frequency establishing the practical limit of the spectrum of a TPW.	Number	[1]
θ	Fraction of the energy of a bandlimited waveform that falls outside the band.	Number	[1]

Symbol	Quantity	Unit	MKS Dimension
λ	Cross-correlation coefficient between two waveforms in a binary system.	Number	[1]
λ_{ij}	Cross-correlation coefficient between the i-th and j-th waveform.	Number	[1]
μ_i	i-th symbol in a symbol alphabet in permutation modulation.	Number	[1]
ρ	Characteristic rate of a system.	Hertz	[s^{-1}]
σ	Power contrast (SNR).	Number or decibel	[1]
σ_{in}	Input power contrast (input SNR).	Number or decibel	[1]
σ_{out}	Output power contrast (output SNR).	Number or decibel	[1]
ϕ_i (t)	i-th orthonormal waveform component function.	*Neutralized	[$V^{1/2}A^{1/}$[illegible]
ω(x)	Weight function in the theory of orthogonal polynomials.	Number	[1]

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[illegible]